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Endomorphism Algebras of Support τ-Tilting Modules*

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Abstract. Let A be a finite dimensional k-algebra and T be a support τ-tilting right A-module. In this note, we give lower and upper bounds for the global dimension of the endomorphism algebra $\mathsf{End}_A(T)$ under some mild conditions. Finally, we give some examples to illustrate that both the upper and lower bounds can be reached.

2020 Mathematics Subject Classification: primary 16E10, 16G10; secondary 18G05, 16E05

Keywords: support τ-tilting module, τ-tilting module, tilting module, global dimension

1 Introduction

In mathematics, specifically representation theory, classical tilting theory describes a way to relate the module categories of two algebras using the so-called tilting modules and associated tilting functors. Classical tilting theory was motivated by the introduction of reflection functors by Bernstein, Gelfand and Ponomarev (see [3]). The tilting theorem, proved by Brenner and Butler, plays an important role in tilting theory. Many of the homological and representation properties of the endomorphism algebra of a tilting module can be deduced from the tilting theorem, for example, the bound of global dimensions, Grothendieck groups and Cartan matrices (see [2]). Recently, Adachi et al. [1] introduced the τ-tilting theory, which is a generalization of classical tilting theory. Motivation for the study of τ-tilting

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theory comes from various sources, and the most important one is mutation of tilting modules. From the viewpoint of mutation, Adachi et al. gave a framework of the mutation for support τ-tilting modules. Later, Treffinger [6] generalized the tilting theorem to the τ-tilting version.

It is well known that τ-tilting modules are precisely sincere support τ-tilting modules (see [1, Proposition 2.2]). In turn, support τ-tilting modules are not necessarily τ-tilting modules over the same algebra. We remark that support τ-tilting modules are closely related to two-term silting complexes (see [1, Theorem 3.2]). Buan and Zhou [4] investigated the global dimension of the endomorphism algebra of a two-term silting complex. Let $P^\bullet$ be a two-term silting complex over the bounded homotopy category $K^b(\mathsf{proj}\, A)$ of finitely generated projective right modules over a finite dimensional k-algebra A. It is known that $H^0(P^\bullet)$ is a support τ-tilting A-module. In general, the endomorphism algebra $\mathsf{End}_A(H^0(P^\bullet))$ is a factor algebra of $\mathsf{End}_{K^b(\mathsf{proj}\, A)}(P^\bullet)$, which is much lager than $\mathsf{End}_A(H^0(P^\bullet))$ (see [1]). Pamela [5] obtained a lower bound of the endomorphism algebra of a τ-tilting A-module via the global dimension of algebra A. However, there is a family of finite dimensional k-algebras of finite global dimension having τ-tilting modules with endomorphism rings of infinite global dimension (see [5]). Hence, it is interesting to find a reasonable homological restriction to give the precise bound of the endomorphism algebra of a support τ-tilting module. Note that each support τ-tilting module can be viewed as a tilting module over some factor algebra. From this point, one may try to deal with the homological dimension by the tilting theorem. In this note, we aim to give the lower and upper bounds for the endomorphism algebra of a support τ-tilting module by combining the τ-tilting theorem with the tilting theorem.

2 Preliminaries

Let k be an algebraically closed field and A a finite dimensional k-algebra. We denote by $\mathsf{mod}\, A$ the category of finitely generated right A-modules. Let τ be the Auslander-Reiten translation in $\mathsf{mod}\, A$. For each A-module M, we denote by $|M|$ the number of non-isomorphic indecomposable direct summands of M. We set $\mathsf{ann}_A\, M$ to be the right annihilator of M. For each M in $\mathsf{mod}\, A$, we define the following subcategories of $\mathsf{mod}\, A$:

$$\mathsf{Fac}\, M = \{X \in \mathsf{mod}\, A \mid \text{there is a surjection } M^{(\lambda)} \to X \text{ for some } \lambda \in \mathbb{Z}\},$$
$$\mathsf{Sub}\, M = \{Y \in \mathsf{mod}\, A \mid \text{there is an injection } Y \to M^{(\mu)} \text{ for some } \mu \in \mathbb{Z}\}.$$

In the following we recall some basic concepts in τ-tilting theory.

Definition 2.1. [1, Definition 0.1] Let M be a module in $\mathsf{mod}\, A$.

(1) M is called *τ-rigid* if $\mathrm{Hom}_A(M, \tau M) = 0$.
(2) M is called *τ-tilting* if it is τ-rigid and $|M| = |A|$.
(3) M is called *support τ-tilting* if it is a τ-tilting $(A/\langle e\rangle)$-module for some idempotent e of A.

Recall that a module $M \in \mathsf{mod}\, A$ is *sincere* if there does not exist a nonzero idempotent e of A that annihilates M. In addition, M is *faithful* if its right annihilator $\mathsf{ann}_A\, M$ vanishes. We collect some useful facts, which we need later.

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Proposition 2.2. [1, Proposition 2.2]
(1) *τ-Tilting modules are precisely sincere support τ-tilting modules.*
(2) *Tilting modules are precisely faithful support τ-tilting modules.*
(3) *Any τ-tilting A-module T is a tilting* $(A/\mathsf{ann}_A T)$*-module.*

Proposition 2.3. [1, Corollary 2.13] *Let T be an A-module. Then T is a support τ-tilting A-module if and only if* $\mathsf{Fac}\, T = \mathsf{Ker}\,\mathrm{Hom}_A(-, \tau T) \cap \mathsf{Ker}\,\mathrm{Hom}_A(eA, -)$, *where e is an idempotent of A such that* $\mathrm{Hom}_A(eA, T) = 0$.

Let T be an A-module and $B = \mathsf{End}_A T$. We set $\mathcal{X} = \{X \in \mathsf{mod}\, B \mid X \otimes_B T = 0\}$ and $\mathcal{Y} = \{Y \in \mathsf{mod}\, B \mid \mathsf{Tor}_1^B(Y, T) = 0\}$.

Theorem 2.4. (τ-tilting theorem) [6, Theorem 2.1] and [7] *Let T be a support τ-tilting A-module. Consider the pair of subcategories* $(\mathsf{Fac}\, T, \mathsf{Sub}\, \tau T)$ *of* $\mathsf{mod}\, A$. *Let* $B = \mathsf{End}_A T$ *and* $C = A/\mathsf{ann}_A T$. *Then there exists a torsion pair* $(\mathcal{X}, \mathcal{Y})$ *in* $\mathsf{mod}\, B$ *such that*
(1) *The functor* $\mathrm{Hom}_A(T, -)\colon \mathsf{Fac}\, T \to \mathcal{Y}$ *is an equivalence of categories with quasi-inverse* $- \otimes_B T\colon \mathcal{Y} \to \mathsf{Fac}\, T$;
(2) *The functor* $\mathsf{Ext}_A^1(T, -)\colon \mathsf{Sub}\, \tau T \to \mathcal{X}$ *is an equivalence of categories with quasi-inverse* $\mathsf{Tor}_1^B(-, T)\colon \mathcal{X} \to \mathsf{Sub}\, \tau T$ *if and only if* $\tau_C T = \tau_A T$.

Lemma 2.5. [5] *Let I be a nilpotent ideal of A. Then the following inequality holds:* $\mathsf{gl.dim}\, A \leq \mathsf{gl.dim}(A/I) + \mathsf{pd}_A(A/I)$.

Proof. Clearly, the statement holds if $\mathsf{pd}_A A/I = \infty$. Now we consider $\mathsf{pd}_A A/I < \infty$. Assume that $\mathsf{pd}_A A/I = n$. For any $M \in \mathsf{mod}\, A^{\mathrm{op}}$, we consider the exact sequence $0 \to M_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, all P_i being projective A^{op}-modules. By dimension shifting, we get the isomorphisms $\mathsf{Tor}_k^A(A/I, M_n) \cong \mathsf{Tor}_{k+n}^A(A/I, M)$ for all $k \geq 1$. Then $\mathsf{Tor}_k^A(A/I, M_n) = 0$ since $\mathsf{pd}_A A/I = n$. By [5, Lemma 2.2], we have $\mathsf{pd}_{A^{\mathrm{op}}} M_n = \mathsf{pd}_{(A/I)^{\mathrm{op}}} M_n/IM_n$. Thus,

$$\begin{aligned}\mathsf{pd}_{A^{\mathrm{op}}} M &\leq n + \mathsf{pd}_{(A/I)^{\mathrm{op}}} M_n/IM_n \\ &\leq \mathsf{pd}_A A/I + \mathsf{gl.dim}\,(A/I)^{\mathrm{op}} \\ &= \mathsf{pd}_A A/I + \mathsf{gl.dim}\, A/I.\end{aligned}$$

So $\mathsf{gl.dim}\, A = \mathsf{gl.dim}\, A^{\mathrm{op}} \leq \mathsf{pd}_A A/I + \mathsf{gl.dim}\, A/I$. □

3 Main Result

The *relative global dimension* of a subclass $\mathcal{C}$ of $\mathsf{mod}\, A$ is defined as

$$\mathsf{rgl.dim}_A \mathcal{C} = \sup\{\mathsf{pd}_A M \mid M \in \mathcal{C}\}.$$

Note that $\mathsf{rgl.dim}_A \mathcal{C} \leq \mathsf{gl.dim}\, A$. Our main result is the following.

Theorem 3.1. *Let T be a support τ-tilting A-module with an idempotent e such that T is a τ-tilting* $(A/\langle e\rangle)$*-module,* $I = \mathsf{ann}_{A/\langle e\rangle} T$, $J = \mathsf{ann}_A T$ *and* $B = \mathsf{End}_A T$. *If* $\mathsf{gl.dim}\, A/J < \infty$ *and* A/J *is a projective A-module, then*

$$\mathsf{gl.dim}\, B \leq \mathsf{rgl.dim}_A \mathsf{Fac}\, T + 1 \leq \mathsf{gl.dim}\, A/J + 1.$$

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On the other hand, if $\mathsf{pd}_{A/\langle e\rangle} I < \infty$, *then*

$$\mathsf{gl.dim}\, A/\langle e\rangle - \mathsf{pd}_{A/\langle e\rangle}(A/J) - 1 \leq \mathsf{gl.dim}\, B.$$

Proof. By [2, Theorem 4.2], we know that $|\mathsf{gl.dim}\, B - \mathsf{gl.dim}\, A/J| \leq 1$ since T is a tilting (A/J)-module. Thus, $\mathsf{gl.dim}\, A/J < \infty$ if and only if $\mathsf{gl.dim}\, B < \infty$.

Assume $\mathsf{gl.dim}\, A/J < \infty$ and that A/J is a projective A-module. Then we have $\mathsf{gl.dim}\, B < \infty$ and $\mathsf{gl.dim}\, B \leq \mathsf{gl.dim}\, A/J + 1$. Next, we prove the first inequality $\mathsf{rgl.dim}_A \mathsf{Fac}\, T + 1 \leq \mathsf{gl.dim}\, A/J + 1$. For any $M \in \mathsf{Fac}\, T$, one can see that $MJ = 0$, and so M is a right (A/J)-module. Since A/J is a projective A-module, we have $\mathsf{pd}_A M \leq \mathsf{pd}_{A/J} M \leq \mathsf{gl.dim}\, A/J$. Thus, $\mathsf{rgl.dim}_A \mathsf{Fac}\, T \leq \mathsf{gl.dim}\, A/J$.

Now, assume that $\mathsf{rgl.dim}_A \mathsf{Fac}\, T = n < \infty$. We divide our proof for the second inequality into two steps.

Step 1. We claim that $\mathsf{pd}_B \mathrm{Hom}_A(T, M) \leq n$ for any $M \in \mathsf{Fac}\, T$.

Since $M \in \mathsf{Fac}\, T$, there is an epimorphism $f\colon T^{(\lambda)} \to M$ for some integer λ. It is well known that $\mathsf{add}\, T$ is a contravariantly finite subcategory of $\mathsf{mod}\, A$. Then there exists an exact sequence

$$0 \longrightarrow K_1 \longrightarrow T_0 \xrightarrow{g_0} M, \tag{1}$$

where g_0 is a right $\mathsf{add}\, T$-approximation of M. Thus, it follows that f factors through g_0, that is, $f = g_0 h$ for some $h \in \mathrm{Hom}_A(T^{(\lambda)}, T_0)$ since g_0 is a right $\mathsf{add}\, T$-approximation. This shows that g_0 is surjective. Since T is τ-rigid, by [1, Lemma 2.6] we have $K_1 \in \mathsf{Ker}\, \mathrm{Hom}_A(-, \tau T)$. Since T is a support τ-tilting module, by Proposition 2.3 we see that there is a projective module P such that $\mathrm{Hom}_A(P, T) = 0$ and $\mathsf{Fac}\, T = \mathsf{Ker}\, \mathrm{Hom}_A(-, \tau T) \cap \mathsf{Ker}\, \mathrm{Hom}_A(P, -)$. Applying $\mathrm{Hom}_A(P, -)$ to the exact sequence (1), we have $K_1 \in \mathsf{Ker}\, \mathrm{Hom}_A(P, -)$ since $\mathrm{Hom}_A(P, T) = 0$. This implies $K_1 \in \mathsf{Fac}\, T$ since $\mathsf{Fac}\, T = \mathsf{Ker}\, \mathrm{Hom}_A(-, \tau T) \cap \mathsf{Ker}\, \mathrm{Hom}_A(P, -)$. Therefore, we have a short exact sequence of A-modules $0 \to K_1 \to T_0 \to M \to 0$ with $K_1 \in \mathsf{Fac}\, T$.

Repeat the above process for K_1, and we have the following exact sequence of A-modules with all $T_i \in \mathsf{add}\, T$:

$$0 \longrightarrow K_{n+1} \longrightarrow T_n \xrightarrow{g_n} T_{n-1} \longrightarrow \cdots \longrightarrow T_0 \xrightarrow{g_0} M \longrightarrow 0. \tag{2}$$

Moreover, by the construction, we see that $K_{i+1} = \mathsf{Ker}\, g_i \in \mathsf{Fac}\, T$ for $0 \leq i \leq n$.

Clearly, $MJ = 0$, and so M is a right (A/J)-module. Thus, the sequence (2) is also an exact sequence of (A/J)-modules.

It is well known that there is a natural inclusion $\mathsf{Ext}^1_{A/J}(M, N) \hookrightarrow \mathsf{Ext}^1_A(M, N)$. By Proposition 2.2, T is a tilting (A/J)-module. Since $K_i \in \mathsf{Ker}\, \mathrm{Hom}_A(-, \tau T)$, we see that $K_i \in \mathsf{Ker}\, \mathsf{Ext}^1_A(T, -)$, and so $K_i \in \mathsf{Ker}\, \mathsf{Ext}^1_{A/J}(T, -)$. It follows that $K_i \in \{Z \in \mathsf{mod}\, A/J \mid \mathsf{Ext}^i_{A/J}(T, Z) = 0, i > 0\}$ since $\mathsf{pd}_{A/J} T \leq 1$.

Applying the functor $\mathrm{Hom}_{A/J}(-, K_{n+1})$ to the sequence (2), we obtain

$$\mathsf{Ext}^{n+1}_{A/J}(M, K_{n+1}) \cong \mathsf{Ext}^1_{A/J}(K_n, K_{n+1}). \tag{3}$$

Since $\mathsf{rgl.dim}_A \mathsf{Fac}\, T = n < \infty$, we have $\mathsf{pd}_A M \leq n$, and so $\mathsf{Ext}^{n+1}_A(M, K_{n+1}) = 0$. Note that A/J is a projective A-module. Accordingly, we notice that the inclusion functor $\mathrm{inc} = \otimes_{A/J} A/J\colon \mathsf{mod}\, A/J \to \mathsf{mod}\, A$ preserves projective modules.

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Hence, it follows that $\mathsf{Ext}^{n+1}_{A/J}(M, K_{n+1}) \cong \mathsf{Ext}^{n+1}_{A}(M, K_{n+1}) = 0$, which means that $\mathsf{Ext}^{1}_{A/J}(K_n, K_{n+1}) = 0$ by the isomorphism (3). Then the short exact sequence $0 \to K_{n+1} \to T_n \to K_n \to 0$ is split. Thus, $K_n \in \mathsf{add}\, T$.

Applying the functor $\mathrm{Hom}_A(T, -)$ to the sequence

$$0 \longrightarrow K_n \longrightarrow T_{n-1} \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_0 \longrightarrow M \longrightarrow 0,$$

we see that $\mathsf{pd}_B \mathrm{Hom}_A(T, M) \leq n$.

Step 2. We prove that $\mathsf{pd}_B Y \leq n+1$ for any $Y \in \mathsf{mod}\, B$.

In fact, by the τ-tilting theorem, there is an equivalence

$$\mathrm{Hom}_A(T, -) : \ \mathsf{Fac}\, T \xrightarrow{\sim} \mathcal{Y} = \{Y \in \mathsf{mod}\, B \mid \mathsf{Tor}^B_1(Y, T) = 0\}.$$

Thus, by Step 1, we see that $\mathsf{pd}_B Y \leq n$ for any $Y \in \mathcal{Y}$. Note that T is a tilting (A/J)-module, and $\mathsf{pd}_{B^{\mathrm{op}}} T \leq 1$ since $B = \mathsf{End}_{A/J}(T)$. Then $\mathsf{Tor}^B_2(Y, T) = 0$ for any $Y \in \mathsf{mod}\, B$. Hence, $\Omega^1_B Y \in \mathcal{Y}$. Now, the claim holds since $\mathsf{pd}\, \Omega^1_B Y = \mathsf{pd}_B Y - 1 \leq n$. Hence, $\mathsf{gl.dim}\, B \leq \mathsf{rgl.dim}_A\, \mathsf{Fac}\, T + 1$. This completes the first statement.

Finally, we turn to prove the second statement. Let $C = A/\langle e\rangle$. Note that T is a τ-tilting C-module. Thus, by Proposition 2.2, T is a tilting C/I-module, where $I = \mathsf{ann}_{A/\langle e\rangle} T$. Then by [2, Theorem 4.2] we have $\mathsf{gl.dim}\, C/I - 1 \leq \mathsf{gl.dim}\, B$. From Lemma 2.5, we get $\mathsf{gl.dim}\, C - \mathsf{pd}_C(C/I) - 1 \leq \mathsf{gl.dim}(C/I) - 1 \leq \mathsf{gl.dim}\, B$. It is easy to see that $C/I \cong A/J$. Indeed, we have a surjective morphism $\phi\colon A \to C/I$ given by $\phi(a) = \overline{a} + I$ for any $a \in A$, where $\overline{a} \in C$. If $\phi(a) = 0$, then $\overline{a} \in I = \mathsf{ann}_{A/\langle e\rangle} T$. Thus, $\overline{a}t = 0$ and so $at = 0$ for any $t \in T$. This means $\mathsf{Ker}\, \phi \subseteq J = \mathsf{ann}_A T$. Note that $\langle e\rangle \subseteq J$. For any $a \in J$, if $a \in \langle e\rangle$, then $\phi(a) = 0$. Otherwise, $0 \neq \overline{a} \in C$. Since $at = 0$ and so $\overline{a}t = 0$ for any $t \in T$, in this case, $\overline{a} \in I$. Then $\phi(a) = 0$. This yields $J \subseteq \mathsf{Ker}\, \phi$. Hence, $\mathsf{Ker}\, \phi = J$.

Therefore, we have $\mathsf{gl.dim}\, C - \mathsf{pd}_C(A/J) - 1 \leq \mathsf{gl.dim}\, B$. □

Remark 3.2. (1) The restriction of homological dimension finiteness on A/J cannot be deleted; see the counterexample [5, Example 2.8]. The assumption of projectiveness on A/J cannot be deleted; see Example 3.5.

(2) For the lower bound, compared with [5, Theorem A], we do not impose the finiteness of global dimension on $A/\langle e\rangle$.

(3) For monomial algebras and special biserial algebras of global dimension two, Pamela [5, Theorems B, C] proved that the global dimension of the endomorphism algebra of any τ-tilting module is always finite. Moreover, for special biserial algebras, he provided an explicit upper bound. In this note, we give the universal upper bound for the endomorphism algebra of any support τ-tilting module.

Recall that if M is a module in $\mathsf{mod}\, A$, then there exists a chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M \tag{4}$$

of submodules of M such that the module M_{j+1}/M_j is simple for $j = 0, 1, \ldots, m-1$. The chain (4) is said to be a *composition series* of M and the simple modules $M_1/M_0, \ldots, M_m/M_{m-1}$ are called the *composition factors* of M.

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Corollary 3.3. *Let T be a support τ-tilting A-module with an idempotent e such that T is a τ-tilting $(A/\langle e\rangle)$-module, $I = \mathsf{ann}_{A/\langle e\rangle} T$, $J = \mathsf{ann}_A T$ and $B = \mathsf{End}_A T$. If $\mathsf{gl.dim}\, A/J < \infty$, A/J is a projective A-module and $\mathsf{pd}_{A/\langle e\rangle} I < \infty$, then*

$$\mathsf{gl.dim}\, A/\langle e\rangle - \mathsf{pd}_{A/\langle e\rangle} A/J - 1 \leq \mathsf{gl.dim}\, B \leq c + 1,$$

where $c = \sup\{\mathsf{pd}_A S \mid S$ is a composition factor of $T\}$.

Proof. Let $d = \sup\{\mathsf{pd}_A S \,|\, S$ is a composition factor of M, where $M \in \mathsf{Fac}\, T\}$ and $c = \sup\{\mathsf{pd}_A S \,|\, S$ is a composition factor of $T\}$. Given a module $M \in \mathsf{Fac}\, T$, using induction on the length of a composition chain of M, we can show $\mathsf{rgl.dim}_A \mathsf{Fac}\, T \leq d$, and so $\mathsf{rgl.dim}_A \mathsf{Fac}\, T = d$. Moreover, since the collection of isomorphic classes of the composition factors of T and that of $\mathsf{Fac}\, T$ are the same, we know that $\mathsf{rgl.dim}_A \mathsf{Fac}\, T = c$. □

Recall that an A-module T is called a support tilting module if it is a tilting $A/\langle e\rangle$-module for some idempotent e. This implies that if T is a support tilting module, then $\mathsf{ann}_{A/\langle e\rangle} T = 0$. Clearly, each support tilting module is a support τ-tilting module. Thus, the next consequence follows directly from Corollary 3.3.

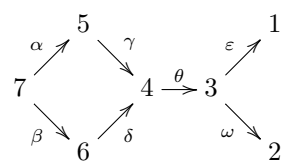
Corollary 3.4. *Let T be a support tilting A-module with an idempotent e such that it is a tilting $A/\langle e\rangle$-module, $J = \mathsf{ann}_A T$ and $B = \mathsf{End}_A T$. If $\mathsf{gl.dim}\, A/J < \infty$ and A/J is a projective A-module, then*

$$\mathsf{gl.dim}\, A/\langle e\rangle - \mathsf{pd}_{A/\langle e\rangle} A/J - 1 \leq \mathsf{gl.dim}\, B \leq c + 1,$$

where $c = \sup\{\mathsf{pd}_A S \mid S$ is a composition factor of $T\}$.

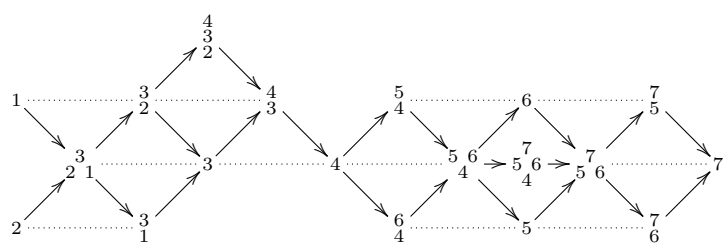
Finally, we give two examples to illustrate that the choice of the upper and lower bounds of the global dimension is reasonable.

Example 3.5. Let A be an algebra given by the bound quiver



with relations $\alpha\gamma = \beta\delta$ and $\theta\varepsilon = \gamma\theta = \delta\theta = 0$. Then $\mathsf{gl.dim}\ A = 4$.

The Auslander-Reiten quiver of $\mathsf{mod}\, A$ can be drawn as follows:



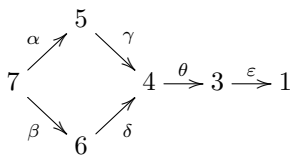
We consider $T = 6 \oplus \begin{smallmatrix} & 7 & \\ 5 & & 6 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 7 \\ 5 \end{smallmatrix} 6 \oplus \begin{smallmatrix} 7 \\ 6 \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$ as a support τ-tilting A-module. It is

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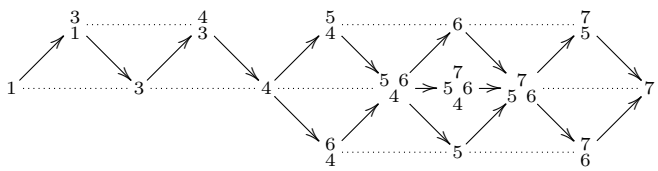
easy to see that $Te_2 = 0$ and

$$\mathsf{Fac}\, T = \left\{ 6\,,\ {}_{4}^{5}6\,,\ {}_{5}^{7}6\,,\ \frac{7}{6}\,,\ \frac{7}{5}\,,\ 7\,,\ 3\,,\ \frac{3}{1} \right\}.$$

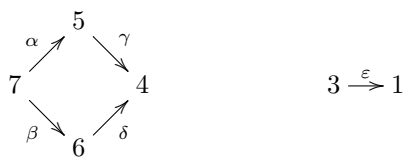
Hence, $\mathsf{rgl.dim}_A \mathsf{Fac}\, T = 4$. Morover, $\mathsf{ann}_A T = \langle e_2, \theta \rangle$ and $\mathsf{ann}_{A/\langle e_2 \rangle} T = \langle \theta \rangle$. One can see that the bound quiver of $A/\langle e_2 \rangle$ is given by



with relations $\alpha\gamma = \beta\delta$ and $\theta\varepsilon = \gamma\theta = \delta\theta = 0$. Then $\mathsf{gl.dim}\, A/\langle e_2 \rangle = 4$. The Auslander-Reiten quiver of $\mathsf{mod}\, A/\langle e_2 \rangle$ can be drawn as follows:

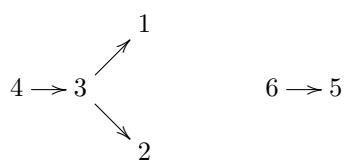


One can obtain $A/\mathsf{ann}_A T = {}_{4}^{5}6 \oplus \frac{5}{4} \oplus \frac{6}{4} \oplus 4 \oplus \frac{3}{1} \oplus 3$ as an $A/\langle e_2 \rangle$-module, and so $\mathsf{pd}_{A/\langle e_2 \rangle} A/\mathsf{ann}_A T = 2$. Clearly, $A/\mathsf{ann}_A T$ is not a projective A-module. The bound quiver of $A/\mathsf{ann}_A T$ is given by



with relation $\alpha\gamma = \beta\delta$. Then $\mathsf{gl.dim}\, A/\mathsf{ann}_A T = 2 < \mathsf{rgl.dim}_A \mathsf{Fac}\, T$.

Moreover, one can compute that B is a hereditary algebra given by the following quiver:

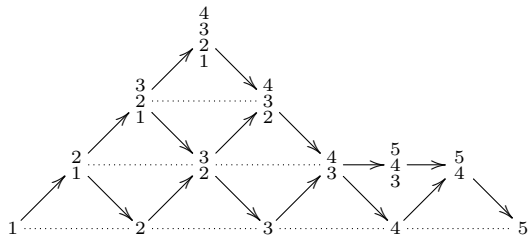


Therefore, $\mathsf{gl.dim}\, A/\langle e_2 \rangle - \mathsf{pd}_{A/\langle e_2 \rangle} A/\mathsf{ann}_A T - 1 = \mathsf{gl.dim}\, B$.

Example 3.6. Let A be an algebra given by the bound quiver

$$5 \xrightarrow{\alpha} 4 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 2 \longrightarrow 1$$

with relation $\alpha\beta\gamma = 0$. Then $\mathsf{gl.dim}\, A = 2$. The Auslander-Reiten quiver of $\mathsf{mod}\, A$ can be drawn as follows:



We consider $T = {}_1 \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus {}_3$ as a support tilting A-module. It is not difficult to get

$$\mathsf{Fac}\, T = \left\{ {}_1\,,\, \begin{smallmatrix}2\\1\end{smallmatrix},\, \begin{smallmatrix}3\\2\\1\end{smallmatrix},\, \begin{smallmatrix}3\\2\end{smallmatrix},\, {}_3 \right\}, \quad \mathsf{ann}_A\, T = \langle e_5, e_4 \rangle.$$

Let $e = e_4 + e_5$. Then $\mathsf{gl.dim}\, A/\mathsf{ann}_A\, T = \mathsf{rgl.dim}\, \mathsf{Fac}\, T = 1$, $A/\mathsf{ann}_A\, T$ is a projective A-module and

$$\mathsf{pd}_{A/\langle e\rangle}\, A/\mathsf{ann}_A\, T = \mathsf{pd}_{A/\langle e\rangle}\, \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus {}_1 = 0.$$

Moreover, it is easy to compute that the bound quiver of B is given by

$$3 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 1, \qquad \alpha\beta = 0.$$

Then $\mathsf{gl.dim}\, A/\langle e\rangle - \mathsf{pd}_{A/\langle e\rangle}\, A/\mathsf{ann}_A\, T - 1 = 0 < \mathsf{gl.dim}\, B = 2 = \mathsf{rgl.dim}_A\, \mathsf{Fac}\, T + 1$.

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Rerefences

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