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One-sided *n*-suspended categories

By

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Abstract

For an integer n >= 3, we introduce a simultaneous generalization of (n - 2)-exact categories and n-angulated categories, referred to as one-sided n-suspended categories. Notably, one-sided n-angulated categories are specific instances of this structure. We establish a framework for transitioning from these generalized categories to their n-angulated counterparts. Additionally, we present a method for constructing n-angulated quotient categories from Frobenius n-prile categories. Our results unify and extend the previous work of Jasso on n-exact categories, Lin on (. + 2)-angulated categories, and Li on one-sided suspended categories.

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Author Keywords: triangulated category; n-angulated category; exact category; (n-2)-exact category; right n-angulated category; one-sided n-suspended category

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ONE-SIDED n-SUSPENDED CATEGORIES

Jing He, Changsha, Yonggang Hu, Beijing, Panyue Zhou, Changsha

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Abstract. For an integer $n \geqslant 3$, we introduce a simultaneous generalization of $(n-2)$-exact categories and n-angulated categories, referred to as one-sided n-suspended categories. Notably, one-sided n-angulated categories are specific instances of this structure. We establish a framework for transitioning from these generalized categories to their n-angulated counterparts. Additionally, we present a method for constructing n-angulated quotient categories from Frobenius n-prile categories. Our results unify and extend the previous work of Jasso on n-exact categories, Lin on $(n+2)$-angulated categories, and Li on one-sided suspended categories.

Keywords: triangulated category; n-angulated category; exact category; $(n-2)$-exact category; right n-angulated category; one-sided n-suspended category

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1. Introduction

Geiss, Keller, and Oppermann in [2] introduced the notion of n-angulated categories, extending the foundational framework of triangulated categories. Notably, certain subcategories, referred to as $(n-2)$-cluster tilting subcategories within triangulated categories, provide concrete illustrations of n-angulated categories, see [2]. In a more recent development, Jasso in [3] introduced the concepts of $(n-2)$-exact categories and algebraic n-angulated categories. It is worth noting that algebraic n-angulated categories, by definition, are equivalent to the stable categories of Frobenius $(n-2)$-exact categories, representing a natural and broader generalization of

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algebraic triangulated categories. Building upon this foundation, Lin in [5] defined the concept of mutation pairs within n-angulated categories and provided a proof demonstrating that when such a mutation pair exists, the corresponding quotient category can be classified as an n-angulated category.

The main aim of this paper is to present a general framework to understand the n-angulated quotient categories that have been discussed in [3], [5]. Drawing inspiration from the notions of a one-sided n-angulated category introduced by Lin (see [6]) and a one-sided suspended category by Li (see [4]), we propose the concept of one-sided n-suspended categories, encompassing $(n-2)$-exact categories, n-angulated categories, and one-sided n-angulated categories as examples.

Roughly speaking, a right n-suspended category consists of an additive category $\mathcal{A}$ endowed with an additive endofunctor Σ, along with two additive subcategories $\mathscr{X}$ and $\mathcal{C}$ of $\mathcal{A}$, together with a class denoted as $R_n(\mathcal{C},\Sigma)$ of n-Σ-sequences. These sequences must adhere to similar axioms as those found in a right n-angulated category, as discussed in [6]. Notably, right n-angulated categories and $(n-2)$-exact categories fall under the umbrella of right n-suspended categories. For the precise definition of a right n-suspended category, please refer to Definition 3.1. Conversely, a left n-suspended category is defined dually. Our first main result is as follows:

Theorem 1.1 (see Theorem 4.6 for details)**.**

(1) *If $(\mathcal{A}, R_n(\mathcal{C},\Sigma), \mathscr{X})$ is a right n-suspended category, then the quotient category $\mathcal{C}/\mathscr{X}$ is a right n-angulated category.*

(2) *If $(\mathcal{A}, \Omega, L_n(\mathcal{C}), \mathscr{X})$ is a left n-suspended category, then the quotient category $\mathcal{C}/\mathscr{X}$ is a left n-angulated category.*

Let $(\mathcal{A}, R_3(\mathcal{C},\Sigma), \mathscr{X})$ be a right 3-suspended category, and $(\mathcal{A}, L_3(\mathcal{C},\Omega), \mathscr{X})$ be a left 3-suspended category. Li in Definition 5.1 of [4] introduced the concept of a "prile" category ("prile" is an abbreviation of "pre", "right" and "left"). It is represented as $(\mathcal{A}, L_3(\mathcal{C},\Omega), R_3(\mathcal{C},\Sigma), \mathscr{X})$. Based on Li's idea, we introduce the concept of n-prile category, see Definition 5.1.

We use our first major result to provide insights into the construction of n-angulated quotient categories. This encompasses numerous existing constructions of n-angulated quotient categories, as elucidated in Theorem 5.11 of [3] and Theorem 3.7 of [5]. To be more precise, we establish the following:

Theorem 1.2 (see Theorem 5.4 for details))**.** *Let $(\mathcal{A}, L_n(\mathcal{C},\Omega), R_n(\mathcal{C},\Sigma), \mathscr{X})$ be a Frobenius n-prile category. Then the quotient category $\mathcal{C}/\mathscr{X}$ is an n-angulated category.*

This paper is organized as follows: In Section 2, we revisit the definitions of an $(n-2)$-exact category and a right n-angulated category. In Section 3, we introduce the concept of a one-sided n-suspended category and present various constructions of one-sided n-suspended categories, drawing from $(n-2)$-exact categories and one-sided n-angulated categories as illustrative examples. In Section 4, we provide a comprehensive framework for transitioning from a one-sided n-suspended category to a one-sided n-angulated category. In Section 5, we delve into the concept of a Frobenius n-prile category, providing illustrative examples and subsequently proving our second major result.

Conventions. Throughout this paper, we assume that n is an integer greater than or equal to three. Let $\mathcal{A}$ be an additive category. When we refer to $\mathscr{X}$ as a subcategory of $\mathcal{A}$, it is implied that $\mathscr{X}$ is a full subcategory and is closed under isomorphisms, direct sums, and direct summands.

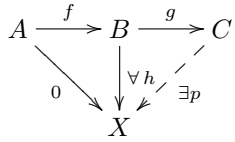
2. Partial one-sided n-angulated categories

2.1. $(n-2)$-exact categories. In this subsection, we recall some definitions of an $(n-2)$-exact category from [3].

Let $\mathcal{A}$ be an additive category and $f\colon A \to B$ a morphism in $\mathcal{A}$. A *weak cokernel* of f is a morphism $g\colon B \to C$ such that for any $X \in \mathcal{A}$ the sequence of abelian groups

$$\mathcal{A}(C,X) \xrightarrow{g^*} \mathcal{A}(B,X) \xrightarrow{f^*} \mathcal{A}(A,X)$$

is exact. Equivalently, g is a weak cokernel of f if $gf = 0$ and for each morphism $h\colon B \to X$ such that $hf = 0$ there exists a (not necessarily unique) morphism $p\colon C \to X$ such that $h = pg$. This description can be represented in the form of the commutative diagram below:



This is also referred to as the factorization property of weak cokernels. Clearly, a weak cokernel g of f is a cokernel of f if and only if g is an epimorphism. The concept of a *weak kernel* is defined dually.

Let $f_1\colon A_1 \to A_2$ be a morphism in $\mathcal{A}$. An $(n-2)$-*cokernel* of f_1 is a sequence

$$(f_2, f_3, \ldots, f_n)\colon A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \ldots \xrightarrow{f_{n-1}} A_n$$

such that for any $B \in \mathcal{A}$ the induced sequence of abelian groups

$$0 \longrightarrow \mathcal{A}(A_n, B) \xrightarrow{(f_{n-1})_*} \mathcal{A}(A_{n-1}, B) \xrightarrow{(f_{n-2})_*} \cdots \xrightarrow{(f_1)_*} \mathcal{A}(A_1, B)$$

is exact. Equivalently, the sequence $(f_2, f_3, \ldots, f_{n-1})$ is an $(n-2)$-cokernel of f_1 if for any $2 \leqslant i \leqslant n-2$ the morphism f_i is a weak cokernel of f_{i-1}, and f_{n-1} is moreover a cokernel of f_{n-2}. In this case, we say the sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \ldots \xrightarrow{f_{n-1}} A_n \tag{2.1}$$

is a *right* $(n-2)$*-exact*. When $n = 3$, the concept of a 1-cokernel is exactly the classical concept of a cokernel. We can define $(n-2)$*-kernel* and left $(n-2)$*-exact sequence* dually. The sequence (2.1) is called $(n-2)$*-exact* if it is both right $(n-2)$-exact and left $(n-2)$-exact.

Let $\mathcal{A}$ be an additive category, $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-2}} A_{n-1}$ a complex and $h_1 \colon A_1 \to B_1$ a morphism in $\mathcal{A}$. An $(n-2)$*-pushout* diagram of $(f_1, f_2, \ldots, f_{n-2})$ along h_1 is a morphism of complexes

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-3}} & A_{n-2} & \xrightarrow{f_{n-2}} & A_{n-1} \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & & & \downarrow{\scriptstyle h_{n-2}} & & \downarrow{\scriptstyle h_{n-1}} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-3}} & B_{n-2} & \xrightarrow{g_{n-2}} & B_{n-1}
\end{array}$$

such that the mapping cone

$$\begin{aligned}
A_1 \xrightarrow{\left[\begin{smallmatrix} -f_1 \\ h_1 \end{smallmatrix}\right]} A_2 \oplus B_1 \xrightarrow{\left[\begin{smallmatrix} -f_2 & 0 \\ h_2 & g_1 \end{smallmatrix}\right]} A_3 \oplus B_2 \xrightarrow{\left[\begin{smallmatrix} -f_3 & 0 \\ h_3 & g_2 \end{smallmatrix}\right]} \cdots \\
\cdots \xrightarrow{\left[\begin{smallmatrix} -f_{n-2} & 0 \\ h_{n-2} & g_{n-3} \end{smallmatrix}\right]} A_{n-1} \oplus B_{n-2} \xrightarrow{[\,h_{n-1}\ g_{n-2}\,]} B_{n-1}
\end{aligned}$$

is a right $(n-2)$-exact.

Let $\mathcal{A}$ be an additive category. If $\mathscr{X}$ is a class of $(n-2)$-exact sequences in $\mathcal{A}$, then we call its members $\mathscr{X}$*-admissible* $(n-2)$*-exact sequences*. Furthermore, if

$$A^\bullet := (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \ldots \xrightarrow{f_{n-1}} A_n)$$

is an $\mathscr{X}$-admissible $(n-2)$-exact sequence, we say that f_1 is an $\mathscr{X}$*-admissible monomorphism* and that f_{n-1} is an $\mathscr{X}$*-admissible epimorphism*. We simply write "admissible" instead of "$\mathscr{X}$-admissible" if the class $\mathscr{X}$ is clear from the context.

Definition 2.1 ([3], Definition 4.2)**.** We say that a morphism $\psi^\bullet\colon A^\bullet \to B^\bullet$ of $(n-2)$-exact sequences in an additive category $\mathcal{A}$ is a *weak isomorphism* if ψ_i and ψ_{i+1} are isomorphisms for some $i \in \{1, 2, \ldots, n\}$ with $\psi_{n+1} := \psi_1$. An $(n-2)$-*exact structure* on $\mathcal{A}$ is a class $\mathscr{X}$ of $(n-2)$-exact sequences in $\mathcal{A}$, closed under weak isomorphisms of $(n-2)$-exact sequences, and which satisfies the following axioms:

(E0) The sequence $0 \to 0 \to \ldots \to 0 \to 0$ is an admissible $(n-2)$-exact sequence.

(E1) The class of admissible monomorphisms is closed under composition.

(E1)$^{\mathrm{op}}$ The class of admissible epimorphisms is closed under composition.

(E2) For any admissible $(n-2)$-exact sequence $A^\bullet$ and any morphism $\varphi\colon A_1 \to B_1$, there exists an $(n-2)$-pushout diagram of $(f_1, f_2, \ldots, f_{n-2})$ along φ such that g_1 is an admissible monomorphism, that is, there exists the following commutative diagram:

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & & & \downarrow{\scriptstyle h_{n-1}} & & \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & B_{n-1} & &
\end{array}$$

(E2)$^{\mathrm{op}}$ For any admissible $(n-2)$-exact sequence $D^\bullet$ and any morphism $\psi\colon C_n \to D_n$, there exists an $(n-2)$-pullback diagram of $(d_2, d_3, \ldots, d_{n-1})$ along ψ such that c_{n-1} is an admissible epimorphism, that is, there exists the following commutative diagram:

$$\begin{array}{ccccccccccc}
C_2 & \xrightarrow{c_2} & C_3 & \xrightarrow{c_3} & \cdots & \xrightarrow{c_{n-2}} & C_{n-1} & \xrightarrow{c_{n-1}} & C_n & & \\
\downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_3} & & & & \downarrow{\scriptstyle e_{n-1}} & & \downarrow{\scriptstyle \psi} & & \\
D_1 & \xrightarrow{d_1} & D_2 & \xrightarrow{d_2} & D_3 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & D_{n-1} & \xrightarrow{d_{n-1}} & D_n
\end{array}$$

An $(n-2)$-*exact category* is a pair $(\mathcal{A}, \mathscr{X})$, where $\mathcal{A}$ is an additive category and $\mathscr{X}$ is an $(n-2)$-exact structure on $\mathcal{A}$. If the class $\mathscr{X}$ is clear from the context, we identify $\mathcal{A}$ with the pair $(\mathcal{A}, \mathscr{X})$.

Let $(\mathcal{A}, \mathscr{X})$ be an $(n-2)$-exact category. An object $I \in \mathcal{A}$ is called *injective* if for every admissible monomorphism $f\colon M \to N$, the sequence of abelian groups

$$\mathcal{A}(N, I) \xrightarrow{f^*} \mathcal{A}(M, I) \to 0$$

is exact. By [3], we say that $(\mathcal{A}, \mathscr{X})$ *has enough injectives* if for every object $M \in \mathcal{A}$ there exist injective objects $I_2, \ldots, I_{n-1}$ and an admissible $(n-2)$-exact sequence

$$M \to I_2 \to \ldots \to I_{n-1} \to N.$$

The notion of *having enough projectives* is defined dually.

We say that an $(n-2)$-exact category $(\mathcal{A}, \mathscr{X})$ is *Frobenius* if it has enough injectives, enough projectives, and if injective and projective objects coincide.

2.2. Right n-angulated categories. Let $\mathcal{A}$ be an additive category with an endofunctor $\Sigma : \mathcal{A} \to \mathcal{A}$. An n-Σ-sequence in $\mathcal{A}$ is a sequence of morphisms

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} \Sigma A_1.$$

Its *left rotation* is the n-Σ-sequence

$$A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4 \xrightarrow{f_4} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} \Sigma A_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma A_2.$$

A *morphism* of n-Σ-sequences is a sequence of morphisms $\varphi = (\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n)$ such that the diagram

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & \Sigma B_1
\end{array}$$

where each row is an n-Σ-sequence, commutes. It is an *isomorphism* if φ_1, φ_2, $\varphi_3, \dots, \varphi_n$ are all isomorphisms in $\mathcal{A}$.

We recall the notion of a right n-angulated category from [6], Definition 2.1. However, the condition (RN4) is slightly different from that in [6], Definition 2.1.

Definition 2.2. A *right n-angulated category* is a triple $(\mathcal{A}, \Sigma, \Theta)$, where $\mathcal{A}$ is an additive category, Σ is an endofunctor of $\mathcal{A}$, and Θ is a class of n-Σ-sequences (whose elements are called right n-angles), which satisfies the following axioms:

(RN1) (a) The class Θ is closed under isomorphisms and direct sums.

(b) For each object $A \in \mathcal{A}$ the trivial sequence

$$A \xrightarrow{1_A} A \to 0 \to \dots \to 0 \to \Sigma A$$

belongs to Θ.

(c) For each morphism $f_1 \colon A_1 \to A_2$ in $\mathcal{C}$, there exists a right n-angle whose first morphism is f_1.

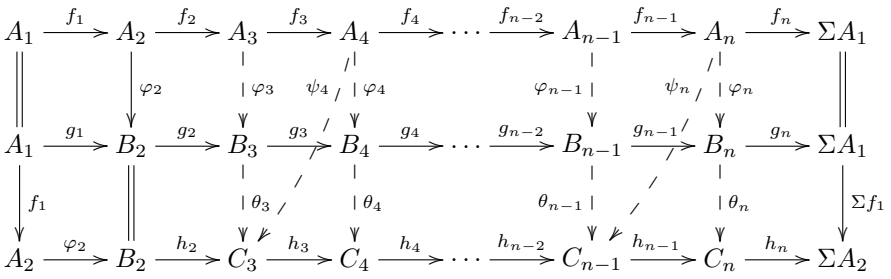
(RN2) If an n-Σ-sequence belongs to Θ, then its left rotation belongs to Θ.

(RN3) Each solid commutative diagram

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \vdots{\scriptstyle \varphi_3} & & & & \vdots{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & \Sigma B_1
\end{array}$$

with rows in Θ can be completed to a morphism of n-Σ-sequences.

(RN4) Given the solid part of the diagram



with commuting squares and with rows in Θ, the dotted morphisms exist such that each square commutes, and the n-Σ-sequence

$$A_3 \xrightarrow{\begin{bmatrix} f_3 \\ \varphi_3 \end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix} -f_4 & 0 \\ \varphi_4 & -g_3 \\ \psi_4 & \theta_3 \end{bmatrix}} A_5 \oplus B_4 \oplus C_3 \xrightarrow{\begin{bmatrix} -f_5 & 0 & 0 \\ -\varphi_5 & -g_4 & 0 \\ \psi_5 & \theta_4 & h_3 \end{bmatrix}}$$

$$A_6 \oplus B_5 \oplus C_4 \xrightarrow{\begin{bmatrix} -f_6 & 0 & 0 \\ \varphi_6 & -g_5 & 0 \\ \psi_6 & \theta_5 & h_4 \end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix} -f_{n-1} & 0 & 0 \\ (-1)^{n+1}\varphi_{n-1} & -g_{n-2} & 0 \\ \psi_{n-1} & \theta_{n-2} & h_{n-3} \end{bmatrix}}$$

$$B_n \oplus C_{n-1} A_n \oplus B_{n-1} \oplus C_{n-2} \xrightarrow{\begin{bmatrix} (-1)^{n+2}\varphi_n & -g_{n-1} & 0 \\ \psi_n & \theta_{n-1} & h_{n-2} \end{bmatrix}}$$

$$B_n \oplus C_{n-1} \xrightarrow{[\,\theta_n \;\; h_n\,]} C_n \xrightarrow{\Sigma f_2 \circ h_n} \Sigma A_3$$

The notion of a *left n-angulated category* is defined dually.

If Σ is an equivalence, and the condition in axiom (RN2) is necessary and sufficient, then the right n-angulated category $(\mathcal{A}, \Sigma, \Theta)$ is an n-angulated category in the sense of Geiss-Keller-Oppermann (see [2], Definition 1.1) and in the sense of Bergh-Thaule, see [1], Theorem 4.4. If $(\mathcal{A}, \Sigma, \Theta)$ is a right n-angulated category, $(\mathcal{A}, \Omega, \Phi)$ is a left n-angulated category, Ω is a quasi-inverse of Σ and $\Theta = \Phi$, then $(\mathcal{A}, \Sigma, \Theta)$ is an n-angulated category.

3. One-sided n-suspended categories

Let $\mathcal{A}$ be an additive category with an additive endofunctor $\Sigma\colon\ \mathcal{A} \to \mathcal{A}$ and $\mathscr{X} \subseteq \mathcal{C}$ two additive subcategories of $\mathcal{A}$. A morphism $f\colon\ A \to B$ with $A, B \in \mathcal{C}$ is called *$\mathscr{X}$-monic* if the induced morphism

$$\mathrm{Hom}_{\mathcal{C}}(f, \mathscr{X})\colon\ \mathrm{Hom}_{\mathcal{C}}(B, \mathscr{X}) \to \mathrm{Hom}_{\mathcal{C}}(A, \mathscr{X})$$

is an epimorphism. An $\mathscr{X}$-*epic* is defined dually. Recall that a morphism $x_M\colon M \to X_M$ is called a left $\mathscr{X}$-approximation of M if x_M is an $\mathscr{X}$-monic and $X_M \in \mathscr{X}$. Dually, a morphism $x_N\colon X_N \to N$ is called a right $\mathscr{X}$-approximation of N if x_N is an $\mathscr{X}$-epic and $X_N \in \mathscr{X}$.

A right n-Σ-sequence $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} \Sigma A_1$ in $\mathcal{A}$ is called a right n-$\mathcal{C}$-*sequence* if $A_3, A_4, \dots, A_n \in \mathcal{C}$ and f_{i+1} is a weak cokernel of f_i for $i = 1, 2, \dots, n-1$. Dually, we can define a left n-$\mathcal{C}$-sequence.

Definition 3.1. A *right n-suspended category* is a triple $(\mathcal{A}, R_n(\mathcal{C}, \Sigma), \mathscr{X})$, where $\mathcal{A}$ is an additive category, Σ is an endofunctor of $\mathcal{A}$, $\mathscr{X} \subseteq \mathcal{C}$ are two additive subcategories of $\mathcal{A}$ and $R_n(\mathcal{C}, \Sigma)$ is a class of right n-$\mathcal{C}$-sequences, which satisfies the following axioms:

(RSN1) (i) The class $R_n(\mathcal{C}, \Sigma)$ is closed under isomorphisms and direct sums.

(ii) For each object $A \in \mathcal{C}$, there exists a right n-$\mathcal{C}$-sequence in $R_n(\mathcal{C}, \Sigma)$

$$A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} B \xrightarrow{y} \Sigma A,$$

where x is a left $\mathscr{X}$-approximation of A and $X_2, X_3, \dots, X_{n-1} \in \mathscr{X}$.

(iii) If $A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} A_n \xrightarrow{y} \Sigma A$ is a right n-$\mathcal{C}$-sequence in $R_n(\mathcal{C}, \Sigma)$, where x is a left $\mathscr{X}$-approximation of A and $X_2, X_3, \dots, X_{n-1} \in \mathscr{X}$, then for each morphism $f\colon A \to B$ in $\mathcal{C}$, there exists a right n-$\mathcal{C}$-sequence in $R_n(\mathcal{C}, \Sigma)$

$$A \xrightarrow{\begin{bmatrix} x \\ f \end{bmatrix}} X_2 \oplus B \to C_3 \to \dots \to C_{n-1} \to C_n \to \Sigma A.$$

(iv) For each morphism $f\colon A \to B$ in $\mathcal{C}$, there exists a right n-$\mathcal{C}$-sequence in $R_n(\mathcal{C}, \Sigma)$

$$A \xrightarrow{\begin{bmatrix} 1 \\ f \end{bmatrix}} A \oplus B \xrightarrow{[f\ -1]} B \to 0 \dots \to 0 \xrightarrow{0} \Sigma A.$$

(RSN2) For each commutative diagram of right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C}, \Sigma)$

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_3} & & & & \downarrow{\scriptstyle a_{n-1}} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \Sigma a} \\
B & \xrightarrow{x_1} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{x_{n-1}} & C & \xrightarrow{x_n} & \Sigma B
\end{array}$$

with $X_2, X_3, \dots, X_{n-1} \in \mathscr{X}$, if a factors through f_1, then b factors through x_{n-1}.

(RSN3) Each solid commutative diagram

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & \Sigma B_1
\end{array}$$

with rows in $R_n(\mathcal{C}, \Sigma)$ can be completed to a morphism of right n-$\mathcal{C}$-sequence.

(RSN4) Given the solid part of the diagram

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & \swarrow{\scriptstyle \psi_4} & \downarrow{\scriptstyle \varphi_4} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & \swarrow{\scriptstyle \psi_n} & \downarrow{\scriptstyle \varphi_n} & & \| \\
A_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & \cdots & \xrightarrow{g_{n-2}} & B_{n-1} & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & \Sigma A_1 \\
\downarrow{\scriptstyle f_1} & & \| & & \downarrow{\scriptstyle \theta_3} & \swarrow & \downarrow{\scriptstyle \theta_4} & & & & \downarrow{\scriptstyle \theta_{n-1}} & \swarrow & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \Sigma f_1} \\
A_2 & \xrightarrow{\varphi_2} & B_2 & \xrightarrow{h_2} & C_3 & \xrightarrow{h_3} & C_4 & \xrightarrow{h_4} & \cdots & \xrightarrow{h_{n-2}} & C_{n-1} & \xrightarrow{h_{n-1}} & C_n & \xrightarrow{h_n} & \Sigma A_2
\end{array}$$

with commuting squares and with rows in $R_n(\mathcal{C}, \Sigma)$, and where f_1, φ_2 are $\mathscr{X}$-monic, the dotted morphisms exist such that each square commutes, and the sequence

$$\begin{array}{l}
A_3 \xrightarrow{\begin{bmatrix} f_3 \\ \varphi_3 \end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix} -f_4 & 0 \\ \varphi_4 & -g_3 \\ \psi_4 & \theta_3 \end{bmatrix}} A_5 \oplus B_4 \oplus C_3 \xrightarrow{\begin{bmatrix} -f_5 & 0 & 0 \\ -\varphi_5 & -g_4 & 0 \\ \psi_5 & \theta_4 & h_3 \end{bmatrix}} \\
A_6 \oplus B_5 \oplus C_4 \xrightarrow{\begin{bmatrix} -f_6 & 0 & 0 \\ \varphi_6 & -g_5 & 0 \\ \psi_6 & \theta_5 & h_4 \end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix} -f_{n-1} & 0 & 0 \\ (-1)^{n-1}\varphi_{n-1} & -g_{n-2} & 0 \\ \psi_{n-1} & \theta_{n-2} & h_{n-3} \end{bmatrix}} \\
A_n \oplus B_{n-1} \oplus C_{n-2} \xrightarrow{\begin{bmatrix} (-1)^{n+2}\varphi_n & -g_{n-1} & 0 \\ \psi_n & \theta_{n-1} & h_{n-2} \end{bmatrix}} \\
B_n \oplus C_{n-1} \xrightarrow{[\,\theta_n \;\; h_n\,]} C_n \xrightarrow{\Sigma f_2 \circ h_n} \Sigma A_3
\end{array}$$

belongs to $R_n(\mathcal{C}, \Sigma)$, where $\begin{bmatrix} f_3 \\ \varphi_3 \end{bmatrix}$ is an $\mathscr{X}$-monic.

Dually, let $\mathcal{A}$ be an additive category with an additive endofunctor $\Omega\colon \mathcal{A} \to \mathcal{A}$ and $\mathscr{X} \subseteq \mathcal{C}$ two subcategories of $\mathcal{A}$. We can define the notion of a *left n-suspended category* $(\mathcal{A}, L_n(\mathcal{C}, \Omega), \mathscr{X})$.

Remark 3.2. In Definition 3.1, when $n = 3$, it is just Definition 3.1 in [4].

We now present constructions of one-sided n-suspended categories by $(n-2)$-exact categories and one-sided n-angulated categories.

3.1. From $(n-2)$-exact categories to one-sided n-suspended categories. First, we present the first construction method of one-sided n-suspended categories by $(n-2)$-exact categories.

Proposition 3.3.

(1) *Let $(\mathcal{A}, \mathscr{E})$ be an $(n-2)$-exact category. Then $(\mathcal{A}, R_n(\mathcal{A},0), \mathcal{A})$ is a right n-suspended category, where $R_n(\mathcal{A},0) = \{A_1 \to A_2 \to A_3 \to \dots A_{n-1} \to A_n \to 0 \mid A_1 \to A_2 \to A_3 \to \dots A_{n-1} \to A_n \in \mathscr{E}\}$.*

(2) *Let $(\mathcal{A}, \mathscr{E})$ be an $(n-2)$-exact category. Then $(\mathcal{A}, L_n(\mathcal{A},0), \mathcal{A})$ is a left n-suspended category, where $L_n(\mathcal{A},0) = \{0 \to A_1 \to A_2 \to A_3 \to \dots A_{n-1} \to A_n \mid A_1 \to A_2 \to A_3 \to \dots A_{n-1} \to A_n \in \mathscr{E}\}$.*

(3) *Let $(\mathcal{A}, \mathscr{E})$ be an $(n-2)$-exact category with enough injectives $\mathscr{I}$. Then $(\mathcal{A}, R_n(\mathcal{A},0), \mathscr{I})$ is a right n-suspended category, where $R_n(\mathcal{A},0) = \{A_1 \to A_2 \to A_3 \to \dots A_{n-1} \to A_n \to 0 \mid A_1 \to A_2 \to A_3 \to \dots A_{n-1} \to A_n \in \mathscr{E}\}$.*

(4) *Let $(\mathcal{A}, \mathscr{E})$ be an $(n-2)$-exact category with enough projectives $\mathscr{P}$. Then $(\mathcal{A}, L_n(\mathcal{A},0), \mathscr{P})$ is a left n-suspended category, where $L_n(\mathcal{A},0) = \{0 \to A_1 \to A_2 \to A_3 \to \dots A_{n-1} \to A_n \mid A_1 \to A_2 \to A_3 \to \dots A_{n-1} \to A_n \in \mathscr{E}\}$.*

P r o o f. (1) By [3], Propositions 4.6 and 4.12, (RSN1) (i) is satisfied. Axioms (RSN1) (ii), (iv) are trivial. Let

$$A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} A_n \to 0$$

be a right n-$\mathcal{A}$-sequence in $R_n(\mathcal{A},0)$, where x is a left $\mathscr{X}$-approximation of A and $X_2, X_3, \dots, X_{n-1} \in \mathscr{X}$. Then there is a right n-$\mathcal{A}$-sequence $A \xrightarrow{-x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} A_n \to 0$. Let $f\colon A \to B$ be a morphism. Since

$$A \xrightarrow{-x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} A_n$$

is an admissible $(n-2)$-exact sequence, by (E2) we get the pushout diagram of $(-x, x_2, \dots, x_{n-2})$ along f:

$$\begin{array}{ccccccccccc}
A & \xrightarrow{-x} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{x_{n-1}} & A_n \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & & & \downarrow{\scriptstyle f_{n-1}} & & \| \\
B & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & B_{n-1} & \xrightarrow{g_{n-1}} & A_n
\end{array}$$

By Proposition 4.8 (2) of [3], we know that the mapping cone of pushout diagram

$$A \xrightarrow{\begin{bmatrix} x \\ f \end{bmatrix}} X_2 \oplus B \xrightarrow{\begin{bmatrix} -x_2 & 0 \\ f_2 & g_1 \end{bmatrix}} X_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -x_3 & 0 \\ f_3 & g_2 \end{bmatrix}} X_4 \oplus B_3 \longrightarrow \dots$$
$$\dots \longrightarrow X_{n-1} \oplus B_{n-2} \xrightarrow{[\, f_{n-1} \;\; g_{n-2} \,]} B_{n-1}$$

is an admissible $(n-2)$-exact sequence. Thus, we have a right n-$\mathcal{A}$-sequence in $R_n(\mathcal{A}, 0)$

$$A \xrightarrow{\begin{bmatrix} x \\ f \end{bmatrix}} X_2 \oplus B \xrightarrow{\begin{bmatrix} -x_2 & 0 \\ f_2 & g_1 \end{bmatrix}} X_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -x_3 & 0 \\ f_3 & g_2 \end{bmatrix}} X_4 \oplus B_3 \longrightarrow \dots$$
$$\dots \longrightarrow X_{n-1} \oplus B_{n-2} \xrightarrow{[\, f_{n-1} \;\; g_{n-2} \,]} B_{n-1} \longrightarrow 0$$

(RSN3) is trivially satisfied. Next we show that (RSN2) holds.

For each commutative diagram of right n-$\mathcal{A}$-sequences

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \longrightarrow & 0 \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_3} & & & & \downarrow{\scriptstyle a_{n-1}} & & \downarrow{\scriptstyle b} & & \\
B & \xrightarrow{x_1} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{x_{n-1}} & C & \longrightarrow & 0
\end{array}$$

with $X_2, X_3, \dots, X_{n-1} \in \mathcal{A}$, assume that a factors through f_1, that is, there exists a morphism $u_2 \colon A_2 \to B$ such that $a = u_2 f_1$. It follows that $(a_2 - x_1 u_2) f_1 = a_2 f_1 - x_1 a = 0$. Since f_2 is a weak cokernel of f_1, there exists a morphism $u_3 \colon A_3 \to X_2$ such that $a_2 - x_1 u_2 = u_3 f_2$. Thus, we have $(a_3 - x_2 u_3) f_2 = a_3 f_2 - x_2 a_2 = 0$. Since f_3 is a weak cokernel of f_2, there exists a morphism $u_4 \colon A_4 \to X_3$ such that $a_3 - x_2 u_3 = u_4 f_3$. Continuing this process, there exists a morphism $u_n \colon A_n \to X_{n-1}$ such that $a_{n-1} - x_{n-2} u_{n-1} = u_n f_{n-1}$. It follows that $b f_{n-1} = x_{n-1} a_{n-1} = x_{n-1}(x_{n-2} u_{n-1} - u_n f_1) = x_{n-1} u_n f_{n-1}$. Since f_{n-1} is an epimorphism, we have $b = x_{n-1} u_n$. This shows that b factors through x_{n-1}.

Next we show that (RSN4) holds. Let the solid part of the diagram

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & & & & & & & & & & & \| \\
A_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & \cdots & \xrightarrow{g_{n-2}} & B_{n-1} & \xrightarrow{g_{n-1}} & B_n & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_1} & & \| & & & & & & & & & & & & \downarrow \\
A_2 & \xrightarrow{\varphi_2} & B_2 & \xrightarrow{h_2} & C_3 & \xrightarrow{h_3} & C_4 & \xrightarrow{h_4} & \cdots & \xrightarrow{h_{n-2}} & C_{n-1} & \xrightarrow{h_{n-1}} & C_n & \longrightarrow & 0
\end{array}$$

with commuting squares and with rows in $R_n(\mathcal{A}, 0)$, and f_1, φ_2 being $\mathcal{A}$-monic, be given. By the factorization property of weak cokernels, there exist morphisms $\varphi_i\colon A_i \to B_i$ $(i = 3, 4, \ldots, n)$ such that we have the following commutative diagram of $(n-2)$-exact sequences:

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n \\
\| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n} \\
A_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & B_{n-1} & \xrightarrow{g_{n-1}} & B_n
\end{array}$$

By [3], Proposition 4.8, we have that

$$\begin{aligned}
A_2 \xrightarrow{\begin{bmatrix} -f_2 \\ \varphi_2 \end{bmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix} -f_4 & 0 \\ \varphi_4 & g_3 \end{bmatrix}} \cdots \\
\cdots \xrightarrow{\begin{bmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{bmatrix}} A_n \oplus B_{n-1} \xrightarrow{\begin{bmatrix} \varphi_n & g_{n-1} \end{bmatrix}} B_n
\end{aligned}$$

is an $(n-2)$-exact sequence. By the factorization property of weak cokernels there exist morphisms $\psi_i\colon A_i \to C_{i-1}$ $(i = 4, 5, \ldots, n)$ and $\theta_j\colon B_j \to C_j$ $(j = 3, 4, \ldots, n)$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
A_2 & \xrightarrow{\begin{bmatrix} -f_2 \\ \varphi_2 \end{bmatrix}} & A_3 \oplus B_2 & \xrightarrow{\begin{bmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{bmatrix}} & A_4 \oplus B_3 & \xrightarrow{\begin{bmatrix} -f_4 & 0 \\ \varphi_4 & g_3 \end{bmatrix}} & \cdots & & \\
\| & & \| & & \downarrow{\scriptstyle \simeq \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} & & & & \\
A_2 & \xrightarrow{\begin{bmatrix} -f_2 \\ \varphi_2 \end{bmatrix}} & A_3 \oplus B_2 & \xrightarrow{\begin{bmatrix} f_3 & 0 \\ \varphi_3 & g_2 \end{bmatrix}} & A_4 \oplus B_3 & \xrightarrow{\begin{bmatrix} f_4 & 0 \\ -\varphi_4 & g_3 \end{bmatrix}} & \cdots & & \\
\| & & \downarrow{\scriptstyle [0\ 1]} & & \downarrow{\scriptstyle [\psi_4\ \theta_3]} & & & & \\
A_2 & \xrightarrow{\varphi_2} & B_2 & \xrightarrow{h_2} & C_3 & \xrightarrow{h_3} & \cdots & &
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \xrightarrow{\begin{bmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{bmatrix}} & A_n \oplus B_{n-1} & \xrightarrow{\begin{bmatrix} \varphi_n & g_{n-1} \end{bmatrix}} & B_n \\
& & \downarrow{\scriptstyle \simeq \begin{bmatrix} (-1)^{n+1} & 0 \\ 0 & 1 \end{bmatrix}} & & \| \\
\cdots & \xrightarrow{\begin{bmatrix} f_{n-1} & 0 \\ (-1)^n \varphi_{n-1} & g_{n-2} \end{bmatrix}} & A_n \oplus B_{n-1} & \xrightarrow{\begin{bmatrix} (-1)^{n+1} \varphi_n & g_{n-1} \end{bmatrix}} & B_n \\
& & \downarrow{\scriptstyle [\psi_n\ \theta_{n-1}]} & & \downarrow{\scriptstyle \theta_n} \\
\cdots & \xrightarrow{h_{n-2}} & C_{n-1} & \xrightarrow{h_{n-1}} & C_n
\end{array}$$

where the second row is an $(n-2)$-exact sequence since it is isomorphic to the first row. By [3], Proposition 4.8, we have that

(3.1)

$$A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -f_3 & 0 \\ -\varphi_3 & -g_2 \\ 0 & 1 \end{bmatrix}} A_4 \oplus B_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -f_4 & 0 & 0 \\ \varphi_4 & -g_3 & 0 \\ \psi_4 & \theta_3 & h_2 \end{bmatrix}} A_5 \oplus B_4 \oplus C_3 \xrightarrow{\begin{bmatrix} -f_5 & 0 & 0 \\ \varphi_5 & -g_4 & 0 \\ \psi_5 & \theta_4 & h_3 \end{bmatrix}} \cdots$$

$$\cdots \xrightarrow{\begin{bmatrix} -f_{n-1} & 0 & 0 \\ (-1)^{n+1}\varphi_{n-1} & -g_{n-2} & 0 \\ \psi_{n-1} & \theta_{n-2} & h_{n-3} \end{bmatrix}} A_n \oplus B_{n-1} \oplus C_{n-2} \xrightarrow{\begin{bmatrix} (-1)^{n+2}\varphi_n & -g_{n-1} & 0 \\ \psi_n & \theta_{n-1} & h_{n-2} \end{bmatrix}}$$

$$B_n \oplus C_{n-1} \xrightarrow{[\theta_n \ h_n]} C_n$$

is an $(n-2)$-exact sequence. The commutative diagram

$$\begin{array}{ccccccc}
A_3 & \xrightarrow{\begin{bmatrix} f_3 \\ \varphi_3 \end{bmatrix}} & A_4 \oplus B_3 & \xrightarrow{\begin{bmatrix} -f_4 & 0 \\ \varphi_4 & -g_3 \\ \psi_4 & \theta_3 \end{bmatrix}} & A_5 \oplus B_4 \oplus C_3 & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \begin{bmatrix} -1 \\ 0 \end{bmatrix}} & & \downarrow{\scriptstyle \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} & & \| & & \\
A_3 \oplus B_2 & \xrightarrow{\begin{bmatrix} -f_3 & 0 \\ -\varphi_3 & -g_2 \\ 0 & 1 \end{bmatrix}} & A_4 \oplus B_3 \oplus B_2 & \xrightarrow{\begin{bmatrix} -f_4 & 0 & 0 \\ \varphi_4 & -g_3 & 0 \\ \psi_4 & \theta_3 & h_2 \end{bmatrix}} & A_5 \oplus B_4 \oplus C_3 & \longrightarrow & \cdots \\
\downarrow{\scriptstyle [-1 \ 0]} & & \downarrow{\scriptstyle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & g_2 \end{bmatrix}} & & \| & & \\
A_3 & \xrightarrow{\begin{bmatrix} f_3 \\ \varphi_3 \end{bmatrix}} & A_4 \oplus B_3 & \xrightarrow{\begin{bmatrix} -f_4 & 0 \\ \varphi_4 & -g_3 \\ \psi_4 & \theta_3 \end{bmatrix}} & A_5 \oplus B_4 \oplus C_3 & \longrightarrow & \cdots
\end{array}$$

shows that

$$A_3 \xrightarrow{\begin{bmatrix} -f_3 \\ \varphi_3 \end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix} -f_4 & 0 \\ -\varphi_4 & -g_3 \\ \psi_4 & \theta_3 \end{bmatrix}} A_5 \oplus B_4 \oplus C_3 \xrightarrow{\begin{bmatrix} -f_5 & 0 & 0 \\ \varphi_5 & -g_4 & 0 \\ \psi_5 & \theta_4 & h_4 \end{bmatrix}} \cdots$$

$$\cdots \xrightarrow{\begin{bmatrix} -f_{n-1} & 0 & 0 \\ (-1)^{n+1}\varphi_{n-1} & -g_{n-2} & 0 \\ \psi_{n-1} & \theta_{n-2} & h_{n-3} \end{bmatrix}} A_n \oplus B_{n-1} \oplus C_{n-2} \xrightarrow{\begin{bmatrix} (-1)^{n+2}\varphi_n & -g_{n-1} & 0 \\ \psi_n & \theta_{n-1} & h_{n-2} \end{bmatrix}}$$

$$B_n \oplus C_{n-1} \xrightarrow{[\theta_n \ h_{n-1}]} C_n$$

is a direct summand of (3.1), thus, it is an $(n-2)$-exact sequence.

Finally, we claim that $\begin{bmatrix} f_3 \\ \varphi_3 \end{bmatrix}$ is an $\mathcal{A}$-monic. Indeed, for any morphism $u\colon A_3 \to X$ with $X \in \mathcal{A}$, since φ_2 is an $\mathcal{A}$-monic, there exists a morphism $v\colon B_2 \to X$ such that $v\varphi_2 = uf_2$. It follows that $vg_1 = v\varphi_2 f_1 = uf_2 f_1 = 0$. So there exists a morphism

$w\colon B_3 \to X$ such that $v = wg_2$. Noting that $(u - w\varphi_3)f_2 = v\varphi_2 - wg_2\varphi_2 = 0$, there exists a morphism $w'\colon A_4 \to X$ such that $u - w\varphi_2 = w'f_3$ and then $u = \begin{bmatrix} w' & w \end{bmatrix} \begin{bmatrix} f_3 \\ \varphi_3 \end{bmatrix}$. This shows that $\begin{bmatrix} f_3 \\ \varphi_3 \end{bmatrix}$ is an $\mathcal{A}$-monic.

(2) It is dual of (1).

(3) It is similar to (1).

(4) It is similar to (2). □

Next, we provide more general construction method by some special subcategories of $(n-2)$-exact categories.

Let $\mathcal{C}$ be an additive category and $R_n(\mathcal{C}, 0)$ the class of right $(n-2)$-exact sequences

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_n \to 0 \quad \text{in } \mathcal{C}.$$

The following lemma can be found in [6].

Lemma 3.4 ([6], Lemma 2.8)**.** *Let*

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n \longrightarrow 0 \\
\| & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & & & \downarrow{\scriptstyle h_{n-1}} & & \downarrow{\scriptstyle h_n} \\
A_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & B_{n-1} & \xrightarrow{g_{n-1}} & B_n \longrightarrow 0
\end{array}$$

be a commutative diagram of right $(n-2)$-exact sequences in $\mathcal{C}$. Then

$$\begin{array}{ccccccccc}
A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n \\
\downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & & & \downarrow{\scriptstyle h_{n-1}} & & \downarrow{\scriptstyle h_n} \\
B_2 & \xrightarrow{g_1} & B_3 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-2}} & B_{n-1} & \xrightarrow{g_{n-1}} & B_n
\end{array}$$

is an $(n-2)$-pushout diagram, that is,

$$\begin{aligned}
A_2 \xrightarrow{\begin{bmatrix} -f_2 \\ h_2 \end{bmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -f_3 & 0 \\ h_3 & g_2 \end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix} -f_4 & 0 \\ h_4 & g_3 \end{bmatrix}} \cdots \\
\cdots \xrightarrow{\begin{bmatrix} -f_{n-1} & 0 \\ h_{n-1} & g_{n-2} \end{bmatrix}} A_n \oplus B_{n-1} \xrightarrow{\begin{bmatrix} h_n & g_{n-1} \end{bmatrix}} B_n \longrightarrow 0
\end{aligned}$$

is a right $(n-2)$-exact sequence in $\mathcal{C}$.

Proposition 3.5. *Let $\mathscr{X}$ be an additive subcategory of $\mathcal{C}$. Assume that the following conditions hold:*

(a) *For each object $A \in \mathcal{C}$, there exists a sequence*

$$A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} C \to 0$$

in $R_n(\mathcal{C}, 0)$, where x is a left $\mathscr{X}$-approximation of A and $X_2, X_3, \dots, X_{n-1} \in \mathscr{X}$.

(b) *If $A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} C \to 0$ is in $R_n(\mathcal{C}, 0)$, where x is a left $\mathscr{X}$-approximation of A and $X_2, X_3, \dots, X_{n-1} \in \mathscr{X}$, then for each morphism $f\colon A \to B$ in $\mathcal{C}$, there exists a sequence*

$$A \xrightarrow{\left[\begin{smallmatrix} x \\ f \end{smallmatrix}\right]} X_2 \oplus B \to C_3 \to \dots \to C_{n-1} \to C_n \to 0$$

in $R_n(\mathcal{C}, 0)$.

Then $(\mathcal{C}, R_n(\mathcal{C}, 0), \mathscr{X})$ is a right n-suspended category.

P r o o f. (RSN1) is trivially satisfied. (RSN2) is similar to the proof of Proposition 3.3 (1). (RSN3) follows from the universal property of n-cokernels. (RSN4) comes from Lemma 3.4 and the proof of Proposition 3.3 (1). □

Dually, let $\mathcal{C}$ be an additive category and $L_n(\mathcal{C}, 0)$ the class of left $(n-2)$-exact sequences

$$0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots \xrightarrow{f_{n-1}} A_n$$

in $\mathcal{C}$. We have the following.

Proposition 3.6. *Let $\mathscr{X}$ be an additive subcategory of $\mathcal{C}$. Suppose the following conditions hold:*

(a) *For each object $A \in \mathcal{C}$, there exists a sequence*

$$0 \to C \xrightarrow{y_1} Y_2 \xrightarrow{y_2} Y_3 \xrightarrow{y_3} \dots \xrightarrow{y_{n-2}} Y_{n-1} \xrightarrow{y} A$$

in $L_n(\mathcal{C}, 0)$, where y is a right $\mathscr{X}$-approximation of A and $Y_2, Y_3, \dots, Y_{n-1} \in \mathscr{X}$.

(b) *If $0 \to C \xrightarrow{y_1} Y_2 \xrightarrow{y_2} Y_3 \xrightarrow{y_3} \dots \xrightarrow{y_{n-2}} Y_{n-1} \xrightarrow{y} A$ is in $L_n(\mathcal{C}, 0)$, where y is a right $\mathscr{X}$-approximation of A and $Y_2, Y_3, \dots, Y_{n-1} \in \mathscr{X}$, then for each morphism $f\colon B \to A$ in $\mathcal{C}$, there exists a sequence*

$$0 \to C_1 \to C_2 \to C_3 \to \dots \to C_{n-2} \to B \oplus Y_{n-1} \xrightarrow{[f\ y]} A$$

in $R_n(\mathcal{C})$.

Then $(\mathcal{C}, L_n(\mathcal{C}, 0), \mathscr{X})$ is a left n-suspended category.

Definition 3.7. Let $(\mathcal{A}, \mathscr{E})$ be an $(n-2)$-exact category and $\mathscr{X} \subseteq \mathcal{C}$ two additive subcategories of $\mathcal{A}$. The subcategory $\mathcal{C}$ is called a *special $\mathscr{X}$-monic closed subcategory* if it satisfies the following conditions:

(a) For each object $A \in \mathcal{C}$, there exists an admissible $(n-2)$-exact sequence

$$A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \ldots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} C,$$

where x is a left $\mathscr{X}$-approximation of A, $X_2, X_3, \ldots, X_{n-1} \in \mathscr{X}$ and $C \in \mathcal{C}$.

(b) If $A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \ldots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} C$ is an admissible $(n-2)$-exact sequence in $\mathcal{C}$, where x is a left $\mathscr{X}$-approximation of A and $X_2, X_3, \ldots, X_{n-1} \in \mathscr{X}$, then for each morphism $f\colon A \to B$ in $\mathcal{C}$, there exists an admissible $(n-2)$-exact sequence

$$A \xrightarrow{\begin{bmatrix} x \\ f \end{bmatrix}} X_2 \oplus B \to C_3 \to \ldots \to C_{n-1} \to C_n,$$

where $C_3, C_4, \ldots, C_n \in \mathcal{C}$.

Dually, we can define the notion of a *special $\mathscr{X}$-epic closed* additive subcategory of $\mathcal{A}$.

Proposition 3.8. *Let $(\mathcal{A}, \mathscr{E})$ be an $(n-2)$-exact category and $\mathscr{X} \subseteq \mathcal{C}$ two additive subcategories of $\mathcal{A}$.*

(1) *If $\mathcal{C}$ is special $\mathscr{X}$-monic closed, then $(\mathcal{A}, R_n(\mathcal{C}, 0), \mathscr{X})$ is a right n-angulated category, where*

$$\begin{aligned} R_n(\mathcal{C}, 0) = \{A \xrightarrow{f} B \to C_3 \to \ldots \to C_n \to 0\colon \\ A \xrightarrow{f} B \to C_3 \to \ldots \to C_n \in \mathscr{E} \text{ and } C_3, \ldots, C_n \in \mathcal{C}\}. \end{aligned}$$

(2) *If $\mathcal{C}$ is special $\mathscr{X}$-epic closed, then $(\mathcal{A}, L_n(\mathcal{C}, 0), \mathscr{X})$ is a left n-angulated category, where*

$$\begin{aligned} L_n(\mathcal{C}, 0) = \{0 \to C_1 \to \ldots \to C_{n-2} \to B \xrightarrow{f} A\colon \\ C_1 \to \ldots \to C_{n-2} \to B \xrightarrow{f} A \in \mathscr{E} \text{ and } C_1, \ldots, C_{n-2} \in \mathcal{C}\}. \end{aligned}$$

P r o o f. Using similar arguments as in the proof of Proposition 3.5, we can show that statement (1) holds by using Proposition 4.8 of [3]. Statement (2) can be showed dually. □

Remark 3.9. If one takes $\mathcal{C} = \mathcal{A}$ and $\mathscr{X} = \mathcal{A}$, then Proposition 3.8 is just Propositions 3.3 (1) and 3.3 (2). If one takes $\mathcal{C} = \mathcal{A}$ with enough injectives or projectives and $\mathscr{X} = \mathscr{P}$ or $\mathscr{I}$, then Proposition 3.8 is just Propositions 3.3 (3) and 3.3 (4).

3.2. From one-sided n-angulated categories to one-sided n-suspended categories. In this section, we construct right n-suspended categories via right n-angulated categories.

Definition 3.10. Let $(\mathcal{A}, \Sigma, \Theta)$ be a right n-angulated category and $\mathscr{X} \subseteq \mathcal{C}$ two additive subcategories of $\mathcal{A}$. The subcategory $\mathcal{C}$ is called *special $\mathscr{X}$-monic closed* if it satisfies the following conditions:

(a) For each object $A \in \mathcal{C}$, there exists a right n-angle

$$A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} C \xrightarrow{x_n} \Sigma A,$$

where x is a left $\mathscr{X}$-approximation of A, $X_2, X_3, \dots, X_{n-1} \in \mathscr{X}$ and $C \in \mathcal{C}$.

(b) If $A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} C \xrightarrow{x_n} \Sigma A$ is a right n-angle in $\mathcal{C}$, where x is a left $\mathscr{X}$-approximation of A and $X_2, X_3, \dots, X_{n-1} \in \mathscr{X}$, then for each morphism $f\colon A \to B$ in $\mathcal{C}$, there exists a right n-angle

$$A \xrightarrow{\left[\begin{smallmatrix} x \\ f \end{smallmatrix}\right]} X_2 \oplus B \to C_3 \to \dots \to C_{n-1} \to C_n \to \Sigma A,$$

where $C_3, C_4, \dots, C_n \in \mathcal{C}$.

(c) If $A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} C \xrightarrow{x_n} \Sigma A$ is a right n-angle, where $X_2, X_3, \dots, X_{n-1} \in \mathscr{X}$ and $C \in \mathcal{C}$, then x_{n-1} is a weak kernel of x_n.

Dually, we can define the notion of a *special $\mathscr{X}$-epic closed* additive subcategory of $\mathcal{A}$.

Remark 3.11. Let $(\mathcal{A}, \Sigma, \Theta)$ be a right n-angulated category. Clearly, $\mathcal{A}$ is special $\mathscr{X}$-monic closed whenever $\mathscr{X} = 0$.

Proposition 3.12. *Let $(\mathcal{A}, \Sigma, \Theta)$ be a right n-angulated category and $\mathscr{X} \subseteq \mathcal{C}$ two additive subcategories of $\mathcal{A}$. If $\mathcal{C}$ is special $\mathscr{X}$-monic closed, then $(\mathcal{A}, R_n(\mathcal{C}, \Sigma), \mathscr{X})$ is a right n-suspended category, where*

$$R_n(\mathcal{C}, \Sigma) = \{A \xrightarrow{f} B \to C_3 \to \dots \to C_n \to \Sigma A \in \Theta\colon\ C_3, \dots, C_n \in \mathcal{C}\}.$$

P r o o f. (RSN1) is trivially satisfied.

(RSN2) For each commutative diagram of right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C}, \Sigma)$

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_3} & & & & \downarrow{\scriptstyle a_{n-1}} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \Sigma a} \\
B & \xrightarrow{x_1} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{x_{n-1}} & C & \xrightarrow{x_n} & \Sigma B
\end{array}$$

with $X_2, X_3, \ldots, X_{n-1} \in \mathscr{X}$, assume that a factors through f_1, that is, there exists a morphism $u\colon A_2 \to B$ such that $a = uf_1$. It follows that $x_n b = \Sigma a \circ f_n = \Sigma u \circ (\Sigma f_1 \circ f_n) = 0$. Since x_{n-1} is a weak kernel of x_n, there exists a morphism $v\colon A_n \to x_{n-1}$ such that $b = x_{n-1}v$. This shows that b factors through x_{n-1}.

(RSN3) follows from (RN3).

(RSN4) follows using similar arguments as in the proof of Proposition 3.3 (1). □

Dually, we have the following.

Proposition 3.13. *Let $(\mathcal{A}, \Omega, \Phi)$ be a left n-angulated category and $\mathscr{X} \subseteq \mathcal{C}$ two additive subcategories of $\mathcal{A}$. If $\mathcal{C}$ is special $\mathscr{X}$-epic closed, then $(\mathcal{A}, L_n(\mathcal{C}, \Omega), \mathscr{X})$ is a left n-suspended category, where*

$$L_n(\mathcal{C}, \Omega) = \{\Omega B \to C_1 \to \ldots \to C_{n-2} \to B \xrightarrow{f} A \in \Phi\colon\ C_1, \ldots, C_{n-2} \in \mathcal{C}\}.$$

Consequently, by Remark 3.11, Propositions 3.12–3.13, we obtain the next result.

Corollary 3.14.

(1) *If $(\mathcal{A}, \Sigma, \Theta)$ is a right n-angulated category, then $(\mathcal{A}, R_n(\mathcal{A}, \Sigma) = \Theta, 0)$ is a right n-suspended category.*

(2) *If $(\mathcal{A}, \Omega, \Phi)$ is a left n-angulated category, then $(\mathcal{A}, L_n(\mathcal{A}, \Omega) = \Phi, 0)$ is a left n-suspended category.*

Let $(\mathcal{A}, \Sigma, \Theta)$ be an n-angulated category and $\mathscr{X}$ a subcategory of $\mathcal{A}$. Recall that $\mathscr{X}$ is called *extension closed* (see [5], Definition 3.6) if for each morphism $x_n\colon X_n \to \Sigma X_1$ with $X_1, X_n \in \mathscr{X}$, there exists an n-angle

$$X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \ldots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} X_n \xrightarrow{x_n} \Sigma X_1$$

with $X_2, X_3, \ldots, X_{n-1} \in \mathscr{X}$.

Lemma 3.15. *Let $(\mathcal{A}, \Sigma, \Theta)$ be an n-angulated category and $\mathscr{X} \subseteq \mathcal{C}$ two additive subcategories of $\mathcal{A}$.*

(1) *If $\mathcal{C}$ is extension closed and for any $A \in \mathcal{C}$, there exists a right n-angle*

$$A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \ldots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} C \xrightarrow{x_n} \Sigma A,$$

where x is a left $\mathscr{X}$-approximation of A, $X_2, X_3, \ldots, X_{n-1} \in \mathscr{X}$ and $C \in \mathcal{C}$, then $\mathcal{C}$ is special $\mathscr{X}$-monic closed.

(2) *If $\mathcal{C}$ is extension closed and for any $A \in \mathcal{C}$, there exists a right n-angle*

$$D \xrightarrow{x'} X_2' \xrightarrow{x_2} X_3' \xrightarrow{x_3'} \ldots \xrightarrow{x_{n-2}'} X_{n-1}' \xrightarrow{x_{n-1}'} A \xrightarrow{x_n'} \Sigma D,$$

where x_n' is a right $\mathscr{X}$-approximation of A, $X_2', X_3', \ldots, X_{n-1}' \in \mathscr{X}$ and $D \in \mathcal{C}$, then $\mathcal{C}$ is special $\mathscr{X}$-epic closed.

P r o o f. (1) It suffices to show that Definition 3.10 (b) holds. Assume that $A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} C \xrightarrow{x_n} \Sigma A$ is a right n-angle in $\mathcal{C}$, where x is a left $\mathscr{X}$-approximation of A and $X_2, X_3, \dots, X_{n-1} \in \mathscr{X}$. Then there is a right n-angle $A \xrightarrow{-x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} C \xrightarrow{-x_n} \Sigma A$ in $\mathcal{C}$.

Let $f\colon A \to B$ be a morphism. For the morphism $C \xrightarrow{d_n := \Sigma f \circ x_n} \Sigma B$ with $B, C \in \mathcal{C}$, since $\mathcal{C}$ is extension closed, there exists an n-angle

$$B \xrightarrow{d_1} D_2 \xrightarrow{d_2} D_3 \xrightarrow{d_3} \dots \xrightarrow{d_{n-2}} D_{n-1} \xrightarrow{d_{n-1}} C \xrightarrow{d_n} \Sigma B$$

with $D_2, D_3, \dots, D_{n-1} \in \mathcal{C}$. Consider the commutative diagram

$$\begin{array}{ccccccccccccc}
\Sigma^{-1}C & \xrightarrow{x_0} & A & \xrightarrow{-x} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{-x_{n-1}} & C \\
\| & & \downarrow{\scriptstyle f} & & & & & & & & & & \| \\
\Sigma^{-1}C & \xrightarrow{d_0} & B & \xrightarrow{d_1} & D_2 & \xrightarrow{d_2} & D_3 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & D_{n-1} & \xrightarrow{d_{n-1}} & C
\end{array}$$

where $x_0 = (-1)^n \Sigma^{-1} x_n$ and $d_0 = (-1)^n \Sigma^{-1} d_n$. By [2], Lemma 4.1, there exist $\varphi_2, \varphi_3, \dots, \varphi_{n-1}$ which give the commutative diagram of n-angles

$$\begin{array}{ccccccccccccc}
\Sigma^{-1}C & \xrightarrow{x_0} & A & \xrightarrow{-x} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{-x_{n-1}} & C \\
\| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \| \\
\Sigma^{-1}C & \xrightarrow{d_0} & B & \xrightarrow{d_1} & D_2 & \xrightarrow{d_2} & D_3 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & D_{n-1} & \xrightarrow{d_{n-1}} & C
\end{array}$$

and the sequence

$$A \xrightarrow{\begin{bmatrix} x \\ f \end{bmatrix}} X_2 \oplus B \to X_3 \oplus C_2 \to \dots \to X_{n-1} \oplus D_{n-2} \to D_{n-1} \to \Sigma A$$

is an n-angle. We observe that $X_3 \oplus C_2, \dots, X_{n-1} \oplus D_{n-2}, D_{n-1} \in \mathcal{C}$. This completes the proof.

(2) It is the dual of (1). □

Example 3.16. Let $(\mathcal{A}, \Sigma, \Theta)$ be an n-angulated category and $\mathscr{X} \subseteq \mathcal{C}$ two additive subcategories of $\mathcal{A}$. If $(\mathcal{C}, \mathcal{C})$ is an $\mathscr{X}$-mutation pair in the sense of Definition 3.1 in [5], that is, for any object $A \in \mathcal{C}$ or $B \in \mathcal{C}$, there exists an n-angle

$$A \xrightarrow{x_1} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} B \xrightarrow{x_n} \Sigma A,$$

where x_1 is a left $\mathscr{X}$-approximation of A and x_{n-1} is a right $\mathscr{X}$-approximation of B, $X_2, X_3, \dots, X_{n-1} \in \mathscr{X}$. Moreover, if $\mathcal{C}$ is extension closed, by Lemma 3.15, then we have that $\mathcal{C}$ is special $\mathscr{X}$-monic closed and $\mathcal{C}$ is special $\mathscr{X}$-epic closed. Thus, $(\mathcal{A}, R_n(\mathcal{C}, \Sigma), \mathscr{X})$ is a right n-suspended category and $(\mathcal{A}, L_n(\mathcal{C}, \Sigma^{-1}), \mathscr{X})$ is a left n-suspended category, where $R_n(\mathcal{C}, \Sigma)$ and $L_n(\mathcal{C}, \Sigma^{-1})$ are constructed in Propositions 3.12 and 3.13.

4. From one-sided n-suspended categories to one-sided n-angulated categories

In this section, we fix a right n-suspended category $(\mathcal{A}, R_n(\mathcal{C}, \Sigma), \mathscr{X})$ and construct the right n-angulated category on the quotient category $\mathcal{C}/\mathscr{X}$.

In order to prove our main theorem of this section, we need to prepare as follows.

Lemma 4.1. *Let*

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & & & \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \Sigma\alpha_1} \\
A & \xrightarrow{d_1} & X_2 & \xrightarrow{d_2} & X_3 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & X_{n-1} & \xrightarrow{d_{n-1}} & B & \xrightarrow{d_n} & \Sigma A
\end{array}$$

be morphisms of right n-$\mathcal{C}$-sequences, where $X_2, X_3, \ldots, X_{n-1} \in \mathscr{X}$ and f_1 is an $\mathscr{X}$-monic. If $\underline{\alpha_1} = 0$, then $\underline{\alpha_n} = 0$ in the quotient category $\mathcal{C}/\mathscr{X}$.

Proof. By the condition $\underline{\alpha_1} = 0$, there exist $X \in \mathscr{X}$, $x_1\colon A_1 \to X$ and $x_2\colon X \to A$ such that $\alpha_1 = x_2x_1$. Since $X \in \mathscr{X}$ and f_1 is $\mathscr{X}$-monic, there exists $x_3\colon A_2 \to X$ such that $x_3f_1 = x_1$. It follows that $\alpha_1 = x_2x_1 = x_2x_3f_1$, that is, α_1 factors through f_1. By (RSN2), α_n factors though d_{n-1}, which implies $\underline{\alpha_n} = 0$. □

Lemma 4.2. *Let $A \xrightarrow{d_1} X_2 \xrightarrow{d_2} X_3 \xrightarrow{d_3} \ldots \xrightarrow{d_{n-2}} X_{n-1} \xrightarrow{d_{n-1}} Y \xrightarrow{d_n} \Sigma A$ and $A \xrightarrow{d'_1} X'_2 \xrightarrow{d_2} X'_3 \xrightarrow{d'_3} \ldots \xrightarrow{d'_{n-2}} X'_{n-1} \xrightarrow{d'_{n-1}} Y' \xrightarrow{d'_n} \Sigma A$ be right $\mathcal{C}$-n-angles in $R_n(\mathcal{C}, \Sigma)$, where $X_2, X_3, \ldots, X_{n-1}, X'_2, X'_3, \ldots, X'_{n-1} \in \mathscr{X}$ and d_1, d'_1 are left $\mathscr{X}$-approximations. Then Y and Y' are isomorphic in $\mathcal{C}/\mathscr{X}$.*

Proof. Since d_1, d'_1 are left $\mathscr{X}$-approximations, we have the following commutative diagram:

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{d_1} & X_1 & \xrightarrow{d_2} & X_2 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & X_{n-1} & \xrightarrow{d_{n-1}} & Y & \xrightarrow{d_n} & \Sigma A \\
\| & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & & & \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_n} & & \| \\
A & \xrightarrow{d'_1} & X'_1 & \xrightarrow{d'_2} & X'_2 & \xrightarrow{d'_3} & \cdots & \xrightarrow{d'_{n-2}} & X'_{n-1} & \xrightarrow{d'_{n-1}} & Y' & \xrightarrow{d'_n} & \Sigma A \\
\| & & \downarrow{\scriptstyle \alpha'_1} & & \downarrow{\scriptstyle \alpha'_2} & & & & \downarrow{\scriptstyle \alpha'_{n-1}} & & \downarrow{\scriptstyle \alpha'_n} & & \| \\
A & \xrightarrow{d_1} & X_1 & \xrightarrow{d_2} & X_2 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & X_{n-1} & \xrightarrow{d_{n-1}} & Y & \xrightarrow{d_n} & \Sigma A
\end{array}$$

By Lemma 4.1, we have $\underline{\alpha'_n} \circ \underline{\alpha_n} = \mathrm{Id}_Y$. Similarly, we can prove that $\underline{\alpha_n} \circ \underline{\alpha'_n} = \mathrm{Id}_{Y'}$. Therefore, Y and Y' are isomorphic in $\mathcal{C}/\mathscr{X}$. □

Let $(\mathcal{A}, R_n(\mathcal{C},\Sigma), \mathscr{X})$ be a right n-suspended category. For any object $A \in \mathcal{C}$, by (RSN1) (ii), there exists a right n-$\mathcal{C}$-sequence

$$A \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \ldots \to X_{n-1} \xrightarrow{f_{n-1}} C \xrightarrow{f_n} \Sigma A_1$$

in $R_n(\mathcal{C},\Sigma)$, where f_1 is a left $\mathscr{X}$-approximation of A and $X_2, X_3, \ldots, X_{n-1} \in \mathscr{X}$. By Lemma 4.2, C is unique up to isomorphism in the quotient category $\mathcal{C}/\mathscr{X}$. So for any $A \in \mathcal{C}$, we fix a right $\mathcal{C}$-n-sequence as above.

For any morphism $f \in \mathcal{C}(A,B)$, there exist φ_i $(i = 2, 3, \ldots, n-1)$ and h which make the following diagram commutative:

$$\begin{array}{ccccccccccccc} A & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & C & \xrightarrow{f_n} & \Sigma A_1 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ B & \xrightarrow{g_1} & X_2' & \xrightarrow{g_2} & X_3' & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & X_{n-1}' & \xrightarrow{g_{n-1}} & D & \xrightarrow{g_n} & \Sigma B_1 \end{array} \tag{4.1}$$

We define a functor $\mathbb{G}\colon \mathcal{C}/\mathscr{X} \longrightarrow \mathcal{C}/\mathscr{X}$ as follows:

$$\mathbb{G}\colon \mathcal{C}/\mathscr{X} \to \mathcal{C}/\mathscr{X}, \quad A \mapsto C, \quad \underline{f} \mapsto \underline{h}.$$

Proposition 4.3. *$\mathbb{G}\colon \mathcal{C}/\mathscr{X} \to \mathcal{C}/\mathscr{X}$ is an additive functor.*

P r o o f. One can easily check that $\mathbb{G}$ satisfies the definition of the additive functor. We only prove that $\mathbb{G}$ is well-defined. Now we assume $f, f' \in \mathcal{C}(A,B)$, $\underline{f} = \underline{f'}$. Then we have the commutative diagram (4.1) and the following commutative diagram, where the rows are right $\mathcal{C}$-n-sequences:

$$\begin{array}{ccccccccccccc} A & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & C & \xrightarrow{f_n} & \Sigma A_1 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \varphi_2'} & & \downarrow{\scriptstyle \varphi_3'} & & & & \downarrow{\scriptstyle \varphi_{n-1}'} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle \Sigma f'} \\ B & \xrightarrow{g_1} & X_2' & \xrightarrow{g_2} & X_3' & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & X_{n-1}' & \xrightarrow{g_{n-1}} & D & \xrightarrow{g_n} & \Sigma B_1 \end{array}$$

Since $\underline{f - f'} = \underline{f} - \underline{f'} = 0$, by Lemma 4.1, we have $\underline{h - h'} = 0$ and then $\underline{h} = \underline{h'}$. This shows that $\mathbb{G}$ is well-defined. □

Construction 4.4. *Assume that $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \ldots \to \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} \Sigma A_1$ is a right n-$\mathcal{C}$-sequence in $R_n(\mathcal{C},\Sigma)$, where f_1 is an $\mathscr{X}$-monic. Thus, we have the following commutative diagram:*

$$\begin{array}{ccccccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\ \| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle h} & & \| \\ A_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & X_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & \mathbb{G}A_1 & \xrightarrow{\alpha_n} & \Sigma A_1 \end{array}$$

Then we have the following n-$\mathbb{G}$-sequence in the quotient category $\mathcal{C}/\mathscr{X}$:

$$A_1 \xrightarrow{\underline{f_1}} A_2 \xrightarrow{\underline{f_2}} A_3 \xrightarrow{\underline{f_3}} \cdots \xrightarrow{\underline{f_{n-1}}} A_n \xrightarrow{\underline{h}} \mathbb{G}A_1. \tag{4.2}$$

We define the standard right n-angles in $\mathcal{C}/\mathscr{X}$ as complexes which are isomorphic to (4.2). *We use Θ' to denote the class of standard right n-angles in $\mathcal{C}/\mathscr{X}$.*

Dually, if $(\mathcal{A}, L_n(\mathcal{C}, \Omega), \mathscr{Y})$ is a left n-suspended category, we can construct an additive endofunctor $\mathbb{H}\colon \mathcal{C}/\mathscr{Y} \to \mathcal{C}/\mathscr{Y}$, and define the corresponding standard left n-angles. We use Φ' to denote the class of standard left n-angles in $\mathcal{C}/\mathscr{Y}$.

Lemma 4.5. *Assume that we have a commutative diagram*

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & \Sigma B_1
\end{array}$$

where the rows are right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C}, \Sigma)$ and f_1, g_1 are $\mathscr{X}$-monic. Then we have the commutative diagram

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{\underline{f_1}} & A_2 & \xrightarrow{\underline{f_2}} & A_3 & \xrightarrow{\underline{f_3}} & \cdots & \xrightarrow{\underline{f_{n-1}}} & A_n & \xrightarrow{\underline{\alpha_n}} & \mathbb{G}A_1 \\
\downarrow{\scriptstyle \underline{\varphi_1}} & & \downarrow{\scriptstyle \underline{\varphi_2}} & & \downarrow{\scriptstyle \underline{\varphi_3}} & & & & \downarrow{\scriptstyle \underline{\varphi_n}} & & \downarrow{\scriptstyle \mathbb{G}\underline{\varphi_1}} \\
B_1 & \xrightarrow{\underline{g_1}} & B_2 & \xrightarrow{\underline{g_2}} & B_3 & \xrightarrow{\underline{g_3}} & \cdots & \xrightarrow{\underline{g_{n-1}}} & B_n & \xrightarrow{\underline{\beta_n}} & \mathbb{G}B_1
\end{array}$$

where the rows are standard right n-angles in $\mathcal{C}/\mathscr{X}$.

P r o o f. Consider the following commutative diagrams, where all three rows in both diagrams are right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C}, \Sigma)$, and $\mathbb{G}\underline{\varphi_1} = \underline{\theta_1}$:

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\| & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & & & \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_n} & & \| \\
A_1 & \xrightarrow{d_1} & X_2 & \xrightarrow{d_2} & X_3 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & X_{n-1} & \xrightarrow{d_{n-1}} & \mathbb{G}A_1 & \xrightarrow{d_n} & \Sigma A_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \alpha'_1} & & \downarrow{\scriptstyle \alpha'_2} & & & & \downarrow{\scriptstyle \alpha'_{n-2}} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
B_1 & \xrightarrow{d'_1} & X'_2 & \xrightarrow{d'_2} & X'_3 & \xrightarrow{d'_3} & \cdots & \xrightarrow{d'_{n-2}} & X'_{n-1} & \xrightarrow{d'_{n-1}} & \mathbb{G}B_1 & \xrightarrow{d'_n} & \Sigma B_1
\end{array}$$

and

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & B_{n-1} & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & \Sigma B_1 \\
\| & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_3} & & & & \downarrow{\scriptstyle \beta_{n-1}} & & \downarrow{\scriptstyle \beta_n} & & \| \\
B_1 & \xrightarrow{d'_1} & X'_2 & \xrightarrow{d'_2} & X'_3 & \xrightarrow{d'_3} & \cdots & \xrightarrow{d'_{n-2}} & X'_{n-1} & \xrightarrow{d'_{n-1}} & \mathbb{G}B_1 & \xrightarrow{d'_n} & \Sigma B_1
\end{array}$$

Since $\underline{\varphi_1 - \varphi_1} = 0$, by Lemma 4.1, we have $\underline{\theta_1 \circ \alpha_n - \beta_n \circ \varphi_n} = 0$ and so $\underline{\theta_1 \circ \alpha_n} = \underline{\beta_n \circ \varphi_n}$. It follows that $\mathbb{G}\underline{\varphi_1} \circ \underline{\alpha_n} = \underline{\beta_n} \circ \underline{\varphi_n}$. Then the following diagram commutes:

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{\underline{f_1}} & A_2 & \xrightarrow{\underline{f_2}} & A_3 & \xrightarrow{\underline{f_3}} & \cdots & \xrightarrow{\underline{f_{n-1}}} & A_n & \xrightarrow{\underline{\alpha_n}} & \mathbb{G}A_1 \\
\downarrow{\scriptstyle \underline{\varphi_1}} & & \downarrow{\scriptstyle \underline{\varphi_2}} & & \downarrow{\scriptstyle \underline{\varphi_3}} & & & & \downarrow{\scriptstyle \underline{\varphi_n}} & & \downarrow{\scriptstyle \mathbb{G}\underline{\varphi_1}} \\
B_1 & \xrightarrow{\underline{g_1}} & B_2 & \xrightarrow{\underline{g_2}} & B_3 & \xrightarrow{\underline{g_3}} & \cdots & \xrightarrow{\underline{g_{n-1}}} & B_n & \xrightarrow{\underline{\beta_n}} & \mathbb{G}B_1
\end{array}$$

□

Now we prove our first main result in this section.

Theorem 4.6.

(1) *If $(\mathcal{A}, R_n(\mathcal{C}, \Sigma), \mathscr{X})$ is a right n-suspended category, then $(\mathcal{C}/\mathscr{X}, \mathbb{G}, \Theta')$ is a right n-angulated category.*

(2) *If $(\mathcal{A}, L_n(\mathcal{C}, \Omega), \mathscr{X})$ is a left n-suspended category, then $(\mathcal{C}/\mathscr{Y}, \mathbb{H}, \Phi')$ is a left n-angulated category.*

P r o o f. (1) We will check the axioms of right n-angulated categories.

It is easy to see that the class of standard right n-angles Θ' is closed under isomorphisms and direct sums. So (RN1)(a) is satisfied.

For any object $A \in \mathcal{C}$, the identity morphism of A is an $\mathscr{X}$-monic. The commutative diagram

$$\begin{array}{ccccccccccc}
A & \xrightarrow{1_A} & A & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \Sigma A \\
\| & & \downarrow{\scriptstyle x_1} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \| \\
A & \xrightarrow{x_1} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{d_{n-2}} & X_{n-1} & \xrightarrow{x_{n-1}} & \mathbb{G}A & \xrightarrow{x_n} & \Sigma A
\end{array}$$

shows that

$$A \xrightarrow{1_A} A \to 0 \to \ldots \to 0 \to \mathbb{G}A$$

belongs to Θ'. Thus (RN1)(b) is satisfied.

For any morphism $f\colon A \to B$ in $\mathcal{C}$, by (RSN1) (ii) and (RSN1) (iii), there exists a right n-$\mathcal{C}$-angle

$$A \xrightarrow{\begin{bmatrix} x \\ f \end{bmatrix}} X_2 \oplus B \to C_3 \to \ldots \to C_{n-1} \to C_n \to \Sigma A,$$

where x is a left $\mathscr{X}$-approximation of A. Noting that $\begin{bmatrix} x \\ f \end{bmatrix}$ is an $\mathscr{X}$-monic, the above right n-$\mathcal{C}$-sequence induces a standard right n-angle

$$A \xrightarrow{\underline{f}} B \to C_3 \to \ldots \to C_{n-1} \to C_n \to \mathbb{G}A.$$

Thus, (RN1) (c) is satisfied.

(RN2) Let $A_1 \xrightarrow{\underline{f_1}} A_2 \xrightarrow{\underline{f_2}} A_3 \xrightarrow{\underline{f_3}} \cdots \xrightarrow{\underline{f_{n-1}}} A_n \xrightarrow{\underline{\alpha_n}} \mathbb{G}A_1$ be a standard right n-angle in $\mathcal{C}/\mathscr{X}$.

For each object $A_1 \in \mathcal{C}$, by (RSN1) (ii), there exists a right n-$\mathcal{C}$-sequence in $R_n(\mathcal{C}, \Sigma)$

$$A_1 \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \ldots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} \mathbb{G}A_1 \xrightarrow{y} \Sigma A_1,$$

where x is a left $\mathscr{X}$-approximation of A and $X_2, X_3, \ldots, X_{n-1} \in \mathscr{X}$. Without loss of generality, we assume that the right n-$\mathcal{C}$-sequence in $R_n(\mathcal{C}, \Sigma)$

$$A_1 \xrightarrow{\begin{bmatrix} x \\ f_1 \end{bmatrix}} X_2 \oplus A_2 \xrightarrow{[\,z\ f_2\,]} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} \Sigma A_1$$

induces the standard right n-angle $A_1 \xrightarrow{\underline{f_1}} A_2 \xrightarrow{\underline{f_2}} A_3 \xrightarrow{\underline{f_3}} \cdots \xrightarrow{\underline{f_{n-1}}} A_n \xrightarrow{\underline{\alpha_n}} \mathbb{G}A_1$, that is, there exists the commutative diagram

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{\begin{bmatrix} x \\ f_1 \end{bmatrix}} & X_2 \oplus A_2 & \xrightarrow{[\,z\ f_2\,]} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\| & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta_3} & & & & \downarrow{\scriptstyle \delta_{n-1}} & & \downarrow{\scriptstyle \alpha_n} & & \| \\
A_1 & \xrightarrow{x} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{x_{n-1}} & \mathbb{G}A_1 & \xrightarrow{y} & \Sigma A_1
\end{array} \tag{4.3}$$

of right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C}, \Sigma)$.

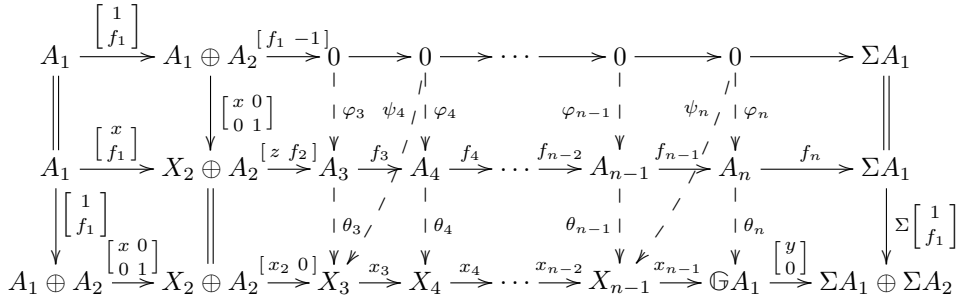
For the morphism $f_1\colon A_1 \to A_2$ in $\mathcal{C}$, by Definition 3.1 (iv), there exists an n-$\mathcal{C}$-sequence in $R_n(\mathcal{C}, \Sigma)$

$$A_1 \xrightarrow{\begin{bmatrix} 1 \\ f_1 \end{bmatrix}} A_1 \oplus A_2 \xrightarrow{[\,f_1\ -1\,]} A_2 \longrightarrow 0 \ldots \longrightarrow 0 \xrightarrow{0} \Sigma A_1.$$

By (RSN1) (i), we have that

$$A_1 \oplus A_2 \xrightarrow{\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}} X_2 \oplus A_2 \xrightarrow{[\,x_2\ 0\,]} X_3 \xrightarrow{x_3} \cdots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} \mathbb{G}A_1 \xrightarrow{\begin{bmatrix} y \\ 0 \end{bmatrix}} \Sigma A_1 \oplus \Sigma A_2$$

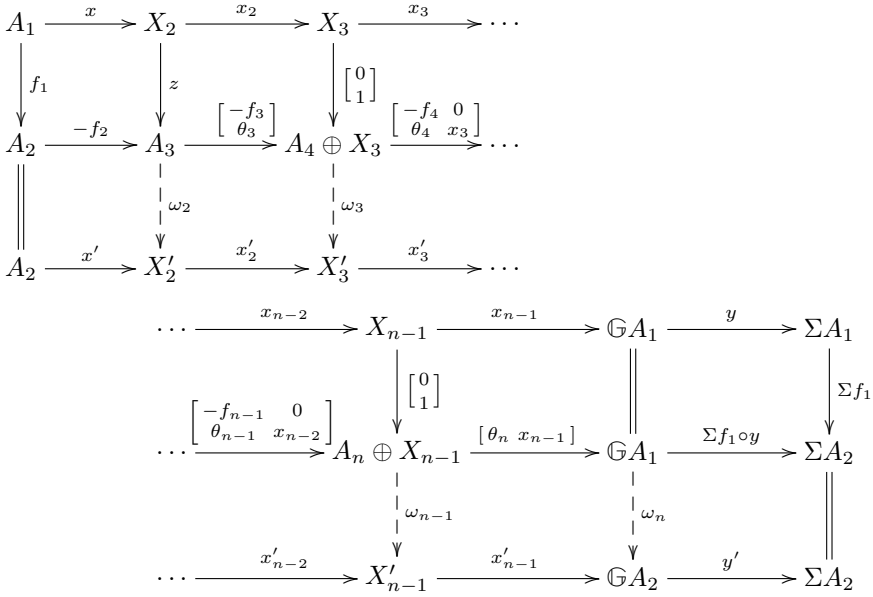
is a right n-$\mathcal{C}$-sequence in $R_n(\mathcal{C},\Sigma)$. By (RSN4), we have the commutative diagram:



of right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C},\Sigma)$ and

$$A_2 \xrightarrow{\varphi_3=-f_2} A_3 \xrightarrow{\begin{bmatrix} -f_3 \\ \theta_3 \end{bmatrix}} A_4 \oplus X_3 \xrightarrow{\begin{bmatrix} -f_4 & 0 \\ \theta_4 & x_3 \end{bmatrix}} \cdots$$
$$\cdots \xrightarrow{\begin{bmatrix} -f_{n-1} & 0 \\ \theta_{n-1} & x_{n-2} \end{bmatrix}} A_n \oplus X_{n-1} \xrightarrow{[\,\theta_n \;\; x_{n-1}\,]} \mathbb{G}A_1 \xrightarrow{\Sigma f_1 \circ y} \Sigma A_2$$

is a right n-$\mathcal{C}$-sequence in $R_n(\mathcal{C},\Sigma)$ with φ_2 being an $\mathscr{X}$-monic. The commutative diagram with rows being right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C},\Sigma)$



implies that $\underline{\omega_n} = \mathbb{G}\underline{f_1}$ and

$$A_2 \xrightarrow{\underline{-f_2}} A_3 \xrightarrow{\underline{-f_3}} A_4 \xrightarrow{\underline{-f_4}} \cdots \xrightarrow{\underline{-f_{n-1}}} A_n \xrightarrow{\underline{\theta_n}} \mathbb{G}A_1 \xrightarrow{\underline{\omega_n}} \mathbb{G}A_2$$

is a standard right n-angle in $\mathcal{C}/\mathscr{X}$. The commutative diagram

$$\begin{array}{ccccccccccccc}
A_2 & \xrightarrow{\underline{-f_2}} & A_3 & \xrightarrow{\underline{-f_3}} & A_4 & \xrightarrow{\underline{-f_4}} & \cdots & \xrightarrow{\underline{-f_{n-1}}} & A_n & \xrightarrow{\underline{\theta_n}} & \mathbb{G}A_1 & \xrightarrow{\underline{\mathbb{G}f_1}} & \mathbb{G}A_2 \\
\| & & \downarrow{\scriptstyle -1} & & \| & & & & \downarrow{\scriptstyle (-1)^n} & & \downarrow{\scriptstyle (-1)^n} & & \| \\
A_2 & \xrightarrow{\underline{f_2}} & A_3 & \xrightarrow{\underline{f_3}} & A_4 & \xrightarrow{\underline{f_4}} & \cdots & \xrightarrow{\underline{f_{n-1}}} & A_n & \xrightarrow{\underline{\theta_n}} & \mathbb{G}A_1 & \xrightarrow{(-1)^n\underline{\mathbb{G}f_1}} & \mathbb{G}A_2
\end{array}$$

shows that

$$A_2 \xrightarrow{\underline{f_2}} A_3 \xrightarrow{\underline{f_3}} A_4 \xrightarrow{\underline{f_4}} \cdots \xrightarrow{\underline{f_{n-1}}} A_n \xrightarrow{\underline{\theta_n}} \mathbb{G}A_1 \xrightarrow{(-1)^n\underline{\mathbb{G}f_1}} \mathbb{G}A_2$$

belongs to Θ'. It remains to show that $\underline{\theta_n} = \underline{\alpha_n}$. Consider the commutative diagram

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{\left[\begin{smallmatrix} x \\ f_1 \end{smallmatrix}\right]} & X_2 \oplus A_2 & \xrightarrow{[\,z\ f_2\,]} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\| & & \downarrow{\scriptstyle [\,1\ 0\,]} & & \downarrow{\scriptstyle \theta_3} & & & & \downarrow{\scriptstyle \theta_{n-1}} & & \downarrow{\scriptstyle \theta_n} & & \| \\
A_1 & \xrightarrow{x} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{x_{n-1}} & \mathbb{G}A_1 & \xrightarrow{y} & \Sigma A_1
\end{array} \tag{4.4}$$

and the commutative diagram (4.3). By Lemma 4.1, we have $\underline{\theta_n} = \underline{\alpha_n}$.

(RN3) Suppose that there exists a commutative diagram

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{\underline{f_1}} & A_2 & \xrightarrow{\underline{f_2}} & A_3 & \xrightarrow{\underline{f_3}} & \cdots & \xrightarrow{\underline{f_{n-1}}} & A_n & \xrightarrow{\underline{\alpha_n}} & \mathbb{G}A_1 \\
\downarrow{\scriptstyle \underline{h_1}} & & \downarrow{\scriptstyle \underline{h_2}} & & & & & & & & \downarrow{\scriptstyle \mathbb{G}\underline{h_1}} \\
B_1 & \xrightarrow{\underline{g_1}} & B_2 & \xrightarrow{\underline{g_2}} & B_3 & \xrightarrow{\underline{g_3}} & \cdots & \xrightarrow{\underline{g_{n-1}}} & B_n & \xrightarrow{\underline{\beta_n}} & \mathbb{G}B_1
\end{array} \tag{4.5}$$

with rows standard right $(n+2)$-angles in $\mathcal{C}/\mathscr{X}$. By assumption, the equality $\underline{h_2} \circ \underline{f_1} = \underline{g_1} \circ \underline{h_1}$ holds, there exist morphisms $u\colon A_1 \to X$ and $v\colon X \to B_2$ with $X \in \mathscr{X}$ such that $h_2 f_1 - g_1 h_1 = vu$. Since f_1 is an $\mathscr{X}$-monic, there exists a morphism $w\colon A_2 \to X$ such that $wf_1 = u$. It follows that $(h_2 - vw)f_1 = h_1 f_1 - vu = g_1 h_1$. By (RSN3), we have the commutative diagram

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2 - vw} & & \| & & & & \| & & \downarrow{\scriptstyle \Sigma h_1} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & \Sigma B_1
\end{array} \tag{4.6}$$

of right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C}, \Sigma)$. By Lemma 4.5 and $\underline{h_2 - vw} = \underline{h_2}$, the diagram (4.5) can be completed to a morphism of right n-angles following from diagram (4.6).

(RN4) Given the solid part of the diagram

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{\underline{f_1}} & A_2 & \xrightarrow{\underline{f_2}} & A_3 & \xrightarrow{\underline{f_3}} & A_4 & \xrightarrow{\underline{f_4}} & \cdots & \xrightarrow{\underline{f_{n-2}}} & A_{n-1} & \xrightarrow{\underline{f_{n-1}}} & A_n & \xrightarrow{\underline{\alpha_n}} & \mathbb{G}A_1 \\
\| & & \downarrow{\scriptstyle \underline{\varphi_2}} & & & & & & & & & & & & \| \\
A_1 & \xrightarrow{\underline{g_1}} & B_2 & \xrightarrow{\underline{g_2}} & B_3 & \xrightarrow{\underline{g_3}} & B_4 & \xrightarrow{\underline{g_4}} & \cdots & \xrightarrow{\underline{g_{n-2}}} & B_{n-1} & \xrightarrow{\underline{g_{n-1}}} & B_n & \xrightarrow{\underline{\beta_n}} & \mathbb{G}A_1 \\
\downarrow{\scriptstyle \underline{f_1}} & & \| & & & & & & & & & & & & \downarrow{\scriptstyle \mathbb{G}\underline{f_1}} \\
A_2 & \xrightarrow{\varphi_2} & B_2 & \xrightarrow{\underline{h_2}} & C_3 & \xrightarrow{\underline{h_3}} & C_4 & \xrightarrow{\underline{h_4}} & \cdots & \xrightarrow{\underline{h_{n-2}}} & C_{n-1} & \xrightarrow{\underline{h_{n-1}}} & C_n & \xrightarrow{\gamma_n} & \mathbb{G}A_2
\end{array}$$

with commuting squares and with rows in Θ', since $\underline{\varphi_2} \circ \underline{f_1} = \underline{g_1}$, there are two morphisms $u\colon A_1 \to X$ and $v\colon X \to B_2$ such that $\varphi_2 f_1 - g_1 = vu$. Since $g_1\colon A_1 \to B_2$ is an $\mathscr{X}$-monic, there exists a morphism $w\colon B_2 \to X$ such that $wg_1 = u$. Thus, $\varphi_2 f_1 = (1 + vw)g_1$. Note that $\underline{g_1} = \underline{(1 + vw)g_1}$. Thus, there is a standard right n-angle

$$A_1 \xrightarrow{\underline{(1+vw)g_1}} B_2 \xrightarrow{\underline{g_2}} B_3 \xrightarrow{\underline{g_3}} \cdots \xrightarrow{\underline{g_{n-1}}} B_n \xrightarrow{\underline{\beta_n}} \mathbb{G}A_1 \in \Theta'.$$

By the construction of Θ', we have a right n-$\mathcal{C}$-sequence

$$A_1 \xrightarrow{(1+vw)g_1} B_2 \xrightarrow{g_2} B_3 \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} B_n \xrightarrow{g_n} \Sigma A_1.$$

By (RSN4), there exist the dotted morphisms such that the each square in

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & \swarrow{\scriptstyle \psi_4} & \downarrow{\scriptstyle \varphi_4} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & \swarrow{\scriptstyle \psi_n} & \downarrow{\scriptstyle \varphi_n} & & \| \\
A_1 & \xrightarrow{(1+vw)g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & \cdots & \xrightarrow{g_{n-2}} & B_{n-1} & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & \Sigma A_1 \\
\downarrow{\scriptstyle f_1} & & \| & & \downarrow{\scriptstyle \theta_3} & \swarrow & \downarrow{\scriptstyle \theta_4} & & & & \downarrow{\scriptstyle \theta_{n-1}} & \swarrow & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \Sigma f_1} \\
A_2 & \xrightarrow{\varphi_2} & B_2 & \xrightarrow{h_2} & C_3 & \xrightarrow{h_3} & C_4 & \xrightarrow{h_4} & \cdots & \xrightarrow{h_{n-2}} & C_{n-1} & \xrightarrow{h_{n-1}} & C_n & \xrightarrow{h_n} & \Sigma A_2
\end{array}$$

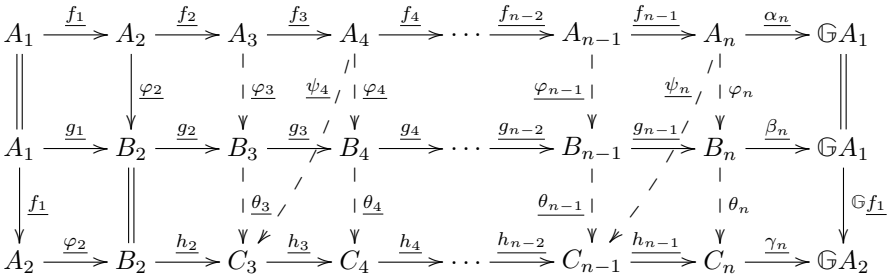
commutes and the sequence

$$A_3 \xrightarrow{\begin{bmatrix} f_3 \\ \varphi_3 \end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix} -f_4 & 0 \\ \varphi_4 & -g_3 \\ \psi_4 & \theta_3 \end{bmatrix}} A_5 \oplus B_4 \oplus C_3 \xrightarrow{\begin{bmatrix} -f_5 & 0 & 0 \\ \varphi_5 & -g_4 & 0 \\ \psi_5 & \theta_4 & h_3 \end{bmatrix}} A_6 \oplus B_5 \oplus C_4$$

$$\xrightarrow{\begin{bmatrix} -f_6 & 0 & 0 \\ \varphi_6 & -g_5 & 0 \\ \psi_6 & \theta_5 & h_4 \end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix} -f_{n-1} & 0 & 0 \\ (-1)^{n+1}\varphi_{n-1} & -g_{n-2} & 0 \\ \psi_{n-1} & \theta_{n-2} & h_{n-3} \end{bmatrix}} A_n \oplus B_{n-1} \oplus C_{n-2}$$

$$\xrightarrow{\begin{bmatrix} (-1)^n\varphi_n & -g_{n-1} & 0 \\ \psi_n & \theta_{n-1} & h_{n-2} \end{bmatrix}} B_n \oplus C_{n-1} \xrightarrow{[\,\theta_n \;\; h_{n-1}\,]} C_n \xrightarrow{\Sigma f_2 \circ h_n} \Sigma A_3$$

is a right n-$\mathcal{C}$-sequence in $R_n(\mathcal{C},\Sigma)$ with $\left[\begin{smallmatrix} f_3 \\ \varphi_3 \end{smallmatrix}\right]$ being an $\mathscr{X}$-monic. Thus, we have the commutative square diagram



and the sequence

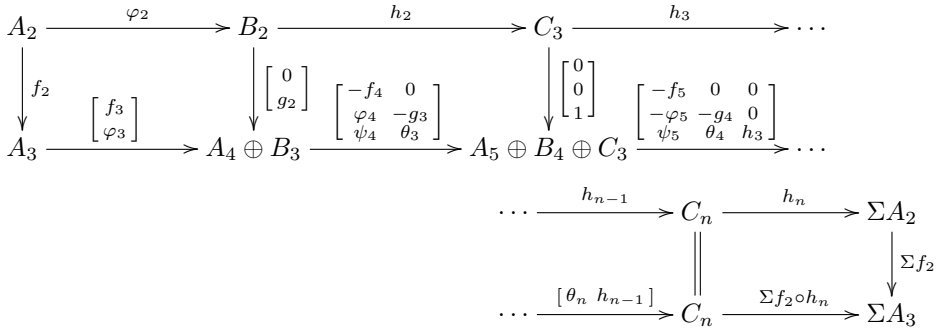
$$A_3 \xrightarrow{\left[\begin{smallmatrix} \underline{f_3} \\ \underline{\varphi_3} \end{smallmatrix}\right]} A_4 \oplus B_3 \xrightarrow{\left[\begin{smallmatrix} \underline{-f_4} & 0 \\ \underline{\varphi_4} & \underline{-g_3} \\ \underline{\psi_4} & \underline{\theta_3} \end{smallmatrix}\right]} A_5 \oplus B_4 \oplus C_3 \xrightarrow{\left[\begin{smallmatrix} \underline{-f_5} & 0 & 0 \\ \underline{\varphi_5} & \underline{-g_4} & 0 \\ \underline{\psi_5} & \underline{\theta_4} & \underline{h_3} \end{smallmatrix}\right]} A_6 \oplus B_5 \oplus C_4$$

$$\xrightarrow{\left[\begin{smallmatrix} \underline{-f_6} & 0 & 0 \\ \underline{\varphi_6} & \underline{-g_5} & 0 \\ \underline{\psi_6} & \underline{\theta_5} & \underline{h_4} \end{smallmatrix}\right]} \cdots \xrightarrow{\left[\begin{smallmatrix} \underline{-f_{n-1}} & 0 & 0 \\ \underline{(-1)^{n+1}\varphi_{n-1}} & \underline{-g_{n-2}} & 0 \\ \underline{\psi_{n-1}} & \underline{\theta_{n-2}} & \underline{h_{n-3}} \end{smallmatrix}\right]} A_n \oplus B_{n-1} \oplus C_{n-2}$$

$$\xrightarrow{\left[\begin{smallmatrix} \underline{(-1)^n\varphi_n} & \underline{-g_{n-1}} & 0 \\ \underline{\psi_n} & \underline{\theta_{n-1}} & \underline{h_{n-2}} \end{smallmatrix}\right]} B_n \oplus C_{n-1} \xrightarrow{\left[\begin{smallmatrix} \underline{\theta_n} & \underline{h_{n-1}} \end{smallmatrix}\right]} C_n \xrightarrow{\underline{\lambda}} \mathbb{G}A_3$$

is a standard right n-angle in $\mathcal{C}/\mathscr{X}$. It suffices to show that $\underline{\lambda} = \mathbb{G}\underline{f_2} \circ \gamma_n$.

Consider the commutative diagram



of right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C},\Sigma)$. By Lemma 4.5, we have $\underline{\lambda} = \mathbb{G}\underline{f_2} \circ \gamma_n$.

(2) It is the dual of (1). □

Remark 4.7. In Theorem 4.6, when $n = 3$, it is just Theorem 3.6 in [4].

5. Frobenius n-Prile categories

In order to generalize the result of Theorem 5.5 in [4] to the version of n-angulated categories, we introduce the notion of a Frobenius n-prile category, which extends the notion of Frobenius prile category defined by Li, see [4].

Definition 5.1. Let $(\mathcal{A}, R_n(\mathcal{C},\Sigma), \mathscr{X})$ be a right n-suspended category and $(\mathcal{A}, L_n(\mathcal{C},\Omega), \mathscr{X})$ a left n-suspended category. The quadruple $(\mathcal{A}, L_n(\mathcal{C},\Omega), R_n(\mathcal{C},\Sigma), \mathscr{X})$ is called a *Frobenius n-prile category* if it satisfies the following conditions:

(a) (Ω,Σ) is an adjoint pair with ψ as the adjunction isomorphism.

(b) If $\Omega A_n \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} A_3 \xrightarrow{f_4} \dots \xrightarrow{f_n} A_n$ is in $L_n(\mathcal{C},\Omega)$ with $A_2,\dots,A_n \in \mathcal{C}$, then

$$A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} A_3 \xrightarrow{f_4} \cdots \xrightarrow{f_n} A_n \xrightarrow{\psi_{A_n,A_1}(f_1)} \Sigma A_1$$

is in $R_n(\mathcal{C},\Sigma)$. Conversely, if $A_1 \xrightarrow{g_1} A_2 \xrightarrow{g_2} A_3 \xrightarrow{g_3} \dots \xrightarrow{g_{n-1}} A_n \xrightarrow{g_n} \Sigma A_1$ is in $R_n(\mathcal{C},\Sigma)$ with $A_1,\dots,A_{n-1} \in \mathcal{C}$, then

$$\Omega A_n \xrightarrow{\psi^{-1}_{A_n,A_1}(g_n)} A_1 \xrightarrow{g_1} A_2 \xrightarrow{g_2} A_3 \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} A_n$$

is in $L_n(\mathcal{C},\Omega)$.

(c) For each object $A \in \mathcal{C}$, x_{n-1} is a left $\mathscr{X}$-approximation of A in the fixed right n-$\mathcal{C}$-sequence

$$A \xrightarrow{x} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \dots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} B \xrightarrow{y} \Sigma A,$$

where $X_2,\dots,X_{n-1} \in \mathscr{X}$ and x'_1 is a right $\mathscr{X}$-approximation of C in the fixed left n-$\mathcal{C}$-sequence

$$\Omega A \xrightarrow{y'} C \xrightarrow{x'_1} X'_2 \xrightarrow{x'_2} X'_3 \xrightarrow{x'_3} \dots \xrightarrow{x'_{n-2}} X'_{n-1} \xrightarrow{x'_{n-1}} A,$$

where $X'_2,\dots,X'_{n-1} \in \mathscr{X}$.

Example 5.2.

(1) Let $\mathcal{B}$ be a Frobenius $(n-2)$-exact category, and $\mathscr{I}$ the subcategory of projective-injective objects of $\mathcal{B}$. Then $(\mathcal{B}, L_n(\mathcal{B},0), R_n(\mathcal{B},0), \mathscr{I})$ is a Frobenius n-prile category, where $L_n(\mathcal{B},\Omega)$ and $R_n(\mathcal{B},\Sigma)$ are constructed in Proposition 3.8.

(2) Let $\mathcal{A}$ be an n-angulated category and $\mathscr{X}$ a subcategory of $\mathcal{C}$. If $\mathcal{C}$ is extension closed and $(\mathcal{C},\mathcal{C})$ is an $\mathscr{X}$-mutation pair, then $(\mathcal{A}, L_n(\mathcal{C},\Sigma^{-1}), R_n(\mathcal{C},\Sigma), \mathscr{X})$ is a Frobenius n-prile category, where $L_n(\mathcal{C},\Sigma^{-1})$ and $R_n(\mathcal{C},\Sigma)$ are constructed in Example 3.16.

Lemma 5.3. *Let $(\mathcal{A}, L_n(\mathcal{C},\Omega), R_n(\mathcal{C},\Sigma), \mathscr{X})$ be a Frobenius n-prile category. Each solid commutative diagram*

$$\begin{array}{ccccccccccccccc}
A & \xrightarrow{a} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-3}} & X_{n-2} & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{x_{n-1}} & B & \xrightarrow{b} & \Sigma A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-2}} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \Sigma f} \\
A' & \xrightarrow{a'} & X'_2 & \xrightarrow{x'_2} & X'_3 & \xrightarrow{x'_3} & \cdots & \xrightarrow{x'_{n-3}} & X'_{n-2} & \xrightarrow{x'_{n-2}} & X'_{n-1} & \xrightarrow{x'_{n-1}} & B' & \xrightarrow{b'} & \Sigma A'
\end{array}$$

with rows being right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C},\Sigma)$ can be completed to a morphism of right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C},\Sigma)$.

P r o o f. Consider the commutative diagram

$$\begin{array}{ccccccccccccccc}
A & \xrightarrow{a} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-3}} & X_{n-2} & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{x_{n-1}} & B & \xrightarrow{b} & \Sigma A \\
 & & & & & & & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle g} & & \\
A' & \xrightarrow{a'} & X'_2 & \xrightarrow{x'_2} & X'_3 & \xrightarrow{x'_3} & \cdots & \xrightarrow{x'_{n-3}} & X'_{n-2} & \xrightarrow{x'_{n-2}} & X'_{n-1} & \xrightarrow{x'_{n-1}} & B' & \xrightarrow{b'} & \Sigma A'
\end{array}$$

of right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C},\Sigma)$. By assumption (b) of Definition 5.1, we have the commutative diagram

$$\begin{array}{ccccccccccccccc}
\Omega B & \xrightarrow{\psi^{-1}_{B,A}(b)} & A & \xrightarrow{a} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-3}} & X_{n-2} & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{x_{n-1}} & B \\
\downarrow{\scriptstyle \Omega(g)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-2}} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle g} \\
\Omega B' & \xrightarrow{\psi^{-1}_{B',A'}(b')} & A' & \xrightarrow{a'} & X'_2 & \xrightarrow{x'_2} & X'_3 & \xrightarrow{x'_3} & \cdots & \xrightarrow{x'_{n-3}} & X'_{n-2} & \xrightarrow{x'_{n-2}} & X'_{n-1} & \xrightarrow{x'_{n-1}} & B'
\end{array}$$

of left n-$\mathcal{C}$-sequences in $L_n(\mathcal{C},\Sigma)$, where the existence of f, $\varphi_2, \ldots, \varphi_{n-2}$ arises from the dual statement of (RSN3).

We claim that $\Sigma f \circ b = b'g$. Indeed, by the naturality of $\psi^{-1}_{B',A'}$ in B', we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{A}}(B', \Sigma A') & \xrightarrow{\psi^{-1}_{B',A'}} & \mathrm{Hom}_{\mathcal{A}}(\Omega B', A') \\
\downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle \Omega(g)^*} \\
\mathrm{Hom}_{\mathcal{A}}(B, \Sigma A') & \xrightarrow{\psi^{-1}_{B,A'}} & \mathrm{Hom}_{\mathcal{A}}(\Omega B, A')
\end{array}$$

It follows that for the morphism $b' \colon B' \to \Sigma A'$, we have

$$\psi^{-1}_{B',A'}(b) \circ \Omega(g) = \psi^{-1}_{B,A'}(b'g).$$

In a similar way, by the naturality of $\psi^{-1}_{B,A}$ in B, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(B,\Sigma A) & \xrightarrow{\psi^{-1}_{B,A}} & \mathrm{Hom}_{\mathcal{A}}(\Omega B, A) \\ \downarrow{\scriptstyle (\Sigma f)_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{Hom}_{\mathcal{A}}(B,\Sigma A') & \xrightarrow{\psi^{-1}_{B,A'}} & \mathrm{Hom}_{\mathcal{A}}(\Omega B, A') \end{array}$$

It follows that for the morphism $b\colon\ B \to \Sigma A$, we have

$$\psi^{-1}_{B,A'}(\Sigma f \circ b) = f \circ \psi^{-1}_{B,A}(b).$$

Since

$$\psi^{-1}_{B',A'}(b') \circ \Omega(g) = f \circ \psi^{-1}_{B,A}(b),$$

we have

$$\psi^{-1}_{B,A'}(b'g) = \psi^{-1}_{B,A'}(\Sigma f \circ b)$$

and then $\Sigma f \circ b = b'g$. Thus, we have the commutative diagram

$$\begin{array}{ccccccccccccc} A & \xrightarrow{a} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \cdots & \xrightarrow{x_{n-3}} & X_{n-2} & \xrightarrow{x_{n-2}} & X_{n-1} & \xrightarrow{x_{n-1}} & B & \xrightarrow{b} & \Sigma A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-2}} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \Sigma f} \\ A' & \xrightarrow{a'} & X'_2 & \xrightarrow{x'_2} & X'_3 & \xrightarrow{x'_3} & \cdots & \xrightarrow{x'_{n-3}} & X'_{n-2} & \xrightarrow{x'_{n-2}} & X'_{n-1} & \xrightarrow{x'_{n-1}} & B' & \xrightarrow{b'} & \Sigma A' \end{array}$$

with rows being right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C},\Sigma)$. □

Theorem 5.4. *Let $(\mathcal{A}, L_n(\mathcal{C},\Omega), R_n(\mathcal{C},\Sigma), \mathscr{X})$ be a Frobenius n-prile category. Then the quotient category $\mathcal{C}/\mathscr{X}$ is an n-angulated category.*

P r o o f. (1) By Theorem 4.6, we know that $(\mathcal{C}/\mathscr{X}, \mathbb{G}, \Theta')$ is a right n-angulated category. It suffices to show that $\mathbb{G}$ is an equivalence.

First, we prove that $\mathbb{G}$ is dense.

For any object $A \in \mathcal{C}/\mathscr{X}$, there exists a right n-$\mathcal{C}$-sequence

$$A \xrightarrow{a} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \ldots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} C \xrightarrow{b} \Sigma C,$$

where a is a left $\mathscr{X}$-approximation of A and $X_2, X_3, \ldots, X_{n-1} \in \mathscr{X}$. By the definition of $\mathbb{G}$ and Lemma 4.2, we have $C \simeq \mathbb{G}(A)$. This shows that $\mathbb{G}$ is dense.

Second, we prove that $\mathbb{G}$ is full.

For any morphism $\underline{h}\colon \mathbb{G}A \to \mathbb{G}B$ in $\mathcal{C}/\mathscr{X}$, since $(\mathcal{A}, L_n(\mathcal{C},\Omega), R_n(\mathcal{C},\Sigma), \mathscr{X})$ is a Frobenius n-prile category, there are two right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C},\Sigma)$:

$$A \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} \mathbb{G}A \xrightarrow{f_n} \Sigma A,$$

where f_{n-1} is a right $\mathscr{X}$-approximation and $X_2, X_3, \ldots, X_{n-1} \in \mathscr{X}$,

$$B \xrightarrow{g_1} X_2' \xrightarrow{g_2} X_3' \xrightarrow{g_3} \cdots \xrightarrow{g_{n-2}} X_{n-1}' \xrightarrow{g_{n-1}} \mathbb{G}B \xrightarrow{g_n} \Sigma B,$$

where g_{n-1} is a right $\mathscr{X}$-approximation and $X_2', X_3', \ldots, X_{n-1}' \in \mathscr{X}$. Thus, there exists the commutative diagram of right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C},\Sigma)$

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & \mathbb{G}A & \xrightarrow{f_n} & \Sigma A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\
B & \xrightarrow{g_1} & X_2' & \xrightarrow{g_2} & X_3' & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & X_{n-1}' & \xrightarrow{g_{n-1}} & \mathbb{G}B & \xrightarrow{g_n} & \Sigma B
\end{array}$$

where φ_{n-1} exists since g_{n-1} is a right $\mathscr{X}$-approximation, $f, \varphi_1, \ldots, \varphi_{n-2}$ exist by Lemma 5.3. It follows that $\underline{h} = \mathbb{G}\underline{f}$. This shows that $\mathbb{G}$ is full.

Finally, we prove that $\mathbb{G}$ is faithful.

Let $f\colon A \to B$ be a morphism in $\mathcal{C}$. There exists the commutative diagram of right n-$\mathcal{C}$-sequences in $R_n(\mathcal{C},\Sigma)$

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & \mathbb{G}A & \xrightarrow{f_n} & \Sigma A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\
B & \xrightarrow{g_1} & X_2' & \xrightarrow{g_2} & X_3' & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & X_{n-1}' & \xrightarrow{g_{n-1}} & \mathbb{G}B & \xrightarrow{g_n} & \Sigma B
\end{array}$$

where f_{n-1} and g_{n-1} are right $\mathscr{X}$-approximations. Assume that $\mathbb{G}\underline{f} = \underline{h} = 0$. Then h factors through some object in $\mathscr{X}$. Since g_{n-1} is a right $\mathscr{X}$-approximation of $\mathbb{G}B_1$, we have that h factors through g_{n-1}. Since $(\mathcal{A}, L_n(\mathcal{C},\Omega), \mathscr{X})$ be a left n-suspended category, we know that f factors through f_1. Therefore, $\underline{f} = 0$. This shows that $\mathbb{G}$ is faithful. □

Remark 5.5. In Theorem 5.4, when $n = 3$, it is just Theorem 5.5 in [4]. Moreover, under the situation of Example 5.2, one can reobtain Theorem 5.11 of [3] and Theorem 3.7 of [5] by Theorem 5.4.

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One-sided n-suspended categories

Jing He, Yonggang Hu, Panyue Zhou

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Abstract: For an integer $n\geq3$, we introduce a simultaneous generalization of $(n-2)$-exact categories and n-angulated categories, referred to as one-sided n-suspended categories. Notably, one-sided n-angulated categories are specific instances of this structure. We establish a framework for transitioning from these generalized categories to their n-angulated counterparts. Additionally, we present a method for constructing n-angulated quotient categories from Frobenius n-prile categories. Our results unify and extend the previous work of Jasso on n-exact categories, Lin on $(n+2)$-angulated categories, and Li on one-sided suspended categories.

Keywords: triangulated category; n-angulated category; exact category; $(n-2)$-exact category; right n-angulated category; one-sided n-suspended category
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For an integer $n \geqslant 3$, we introduce a simultaneous generalization of $(n-2)$-exact categories and n-angulated categories, referred to as one-sided n-suspended categories. Notably, one-sided n-angulated categories are specific instances of this structure. We establish a framework for transitioning from these generalized categories to their n-angulated counterparts. Additionally, we present a method for constructing n-angulated quotient categories from Frobenius n-prile categories. Our results unify and extend the previous work of Jasso on n-exact categories, Lin on $(n+2)$-angulated categories, and Li on one-sided suspended categories.

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